Some Contributions to Life Testing Models

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CERTIFICATE

It is certified that the work contained in the thesis titled *"Some Contributions to Life Testing Models"*, by Mr. Ayon Ganguly, has been carried out under our supervision and that this work has not been submitted elsewhere for a degree.

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SYNOPSIS

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Life testing experiments have gained popularity in recent times. The main aim of a life testing experiment is to measure one or more reliability characteristics of the product under consideration. In a very classical form of life testing experiment, certain number of identical items are placed on the test under normal operating conditions and the 'time to failure' of all the items are recorded. One of the main problems of the classical life testing experiment is the long experimental time. To cope with this problem different censoring schemes are proposed in the literature. The most popular censoring schemes are Type-I and Type-II censoring schemes. In Type-I censoring scheme, experimenter reduces the experimental time by terminating the experiment at a pre-fixed time, whereas in Type-II censoring scheme, experiment terminates as soon as a pre-specified number of failures occur. Different type of mixtures of these two basic censoring schemes, known as hybrid censoring schemes, are introduced in the literature by several authors. Progressive censoring schemes are proposed to remove experimental units in working conditions from the experiment before the termination of the experiment. However, there may be few failures or even no failure before the terminating time in the Type-I censoring scheme, which creates difficulty in further statistical analysis. On the other hand, the main drawback of the Type-II censoring scheme is the possibility of long duration of experimental time. The hybrid and progressive censoring schemes inherit one or both of these problems. In the literature different censoring schemes have been studied under different lifetime distributions. In this context, readers are referred to the text by Lawless [93] and the review articles by Balakrishnan and Kundu [28], and Balakrishnan [9].

Accelerated life test is a very useful technique for testing reliability characteristics of durable products. In an accelerated life test the product is tested under one or more extreme environmental conditions which affect the lifetime of the product negatively and thus an accelerated life test helps in obtaining more failures in an affordable time. This type of factors which affect lifetime of the products are called stress factors. Step-stress life test is a special type of accelerated life test and allows the experimenter to change the stress levels during the experiment. However, in most of the cases a step-stress life test is done with censoring schemes. Though different lifetime distributions have been studied in the literature, it may be mentioned that the one-parameter exponential distribution has been explored most extensively. A review of step-stress model under different censoring schemes is provided by Balakrishnan [10].

Childs et al. [49] considered the maximum likelihood estimators of the unknown parameters of a two-parameter exponential model when the data are hybrid Type-I censored. The exact distribution of the maximum likelihood estimators of unknown parameters and of the quantiles were provided in this article. They also discussed different methods for constructing confidence intervals of the unknown parameters and compared them based on extensive simulation study. In Chapter 2, we have addressed the problem of estimation of model parameters of two-parameter exponential distribution in presence of hybrid Type-II censoring scheme. The distributions of maximum likelihood estimators of the scale and the location parameters have been derived. Based on the assumption of the monotonicity of the cumulative distribution function of the maximum likelihood estimator of the scale parameters, approximate confidence interval of the scale parameter has been derived in this dissertation. However, we could not prove this monotonicity assumption formally, the same has been verified on the basis of extensive numerical simulations. Percentile and bias corrected Bootstrap confidence intervals have also been considered. Simulations have been done to judge the performance of the different confidence intervals. Analysis of a data set has been performed for illustrative purpose.

Extensive work has been done on different censoring schemes by several authors. Lawless [93], Balakrishnan [9], Balakrishnan and Kundu [28] and references cited therein may be mentioned in this regard. However, most of the analysis has been performed under the frequentist context and very little attention is paid to Bayesian analysis. Moreover, it is worth mentioning that the analysis of different hybrid and progressive censoring schemes is not very easy even when lifetimes of the experimental units are assumed to have an exponential distribution. Though finding the maximum likelihood estimates are not difficult, construction of the confidence interval involves a numerical computation. It seems that Bayesian approach is a natural choice in this case. Draper and Guttman [62] considered Bayesian analysis of hybrid life tests with one-parameter exponential failure times. In Chapter 3, an attempt has been made to address the Bayesian inference of the unknown parameters of a two-parameter exponential distribution when the data are obtained from different hybrid and progressive censoring schemes. We notice that the Bayes estimate and credible interval of some parametric function cannot be found in explicit form in general. A simulation based procedure has been proposed to compute Bayes estimate as well as to construct credible intervals. Extensive simulation study has been carried out to understand the effectiveness of the proposed procedure. Analysis of a data set has been performed for illustrative purpose.

Simple step-stress models under different censoring schemes are extensively studied based on the assumption that the lifetime of the experimental units follow exponential distributions with different scale parameters at different stress levels. Different models have been proposed to relate cumulative distribution function under different stress levels to that under step-stress pattern by several authors. Among them the most popular one is cumulative exposure model and most of the literature is developed based on this model assumption. Interested readers are referred to the review article by Balakrishnan [10] in this respect. In all these cases the exact distributions of the unknown parameters are obtained, and they can be used to construct exact confidence intervals. However, it is observed that the exact distribution and therefore the construction of associated confidence interval is quite complicated in all these cases. It may be mentioned that although extensive work has been done on step-stress models, not much attention has been paid to develop the inference imposing the order restriction on the mean lifetime of the product at different stress levels, which is a very natural choice for simple step-stress life test. Balakrishnan et al. [13] considered the order restricted inference for step-stress models when lifetimes are independently and exponentially distributed, and the data are Type-I or Type-II censored. It is observed that obtaining the exact joint distribution of the maximum likelihood estimators is not easy. It is not immediate that how this method can be extended for more general censoring schemes. It seems that Bayesian analysis is a natural choice in these cases. Though some work have been done on the Bayesian inference of the step-stress model, see for example Drop et al. [64], Lee and Pan [94], Leu and Shen [95] or Fan et al. [71], none of them dealt with the ordered restricted inference. We have addressed order restricted Bayesian inference of the unknown parameters of a simple step-stress model under different censoring schemes when the lifetimes of the experimental units are assumed to be exponentially distributed in Chapter 4. We have assumed fairly flexible priors on the unknown parameters. It has been observed that in most of the cases the Bayes estimates of the unknown parameters cannot be obtained in explicit form. We have proposed to use the importance sampling technique to compute Bayes estimate and also to construct associated credible interval. Extensive Monte Carlo simulations have been performed to see the effectiveness of the proposed method in case of Type-I censoring scheme. The analysis of two data sets have been performed for illustrative purposes.

Though analysis of one-parameter exponential model in case of simple step-stress set up, under the cumulative exposure model formulation has been performed quite extensively in the literature, not much work has been done in case of two-parameter exponential distribution. In Chapter 5, an attempt has been made to address the same issue. We have analyzed a simple step-stress model based on the assumption that the lifetimes of the experimental units follow two-parameter exponential distribution. The analysis has been performed based on the assumption that the model satisfies cumulative exposure model assumptions, and the data are Type-II censored. One of the justifications for incorporating the location parameter is the presence of possible bias in the experimental data due to calibration. It is observed that the maximum likelihood estimators of the unknown parameters do not always exist. Whenever they exist, they can be obtained in closed form. We have obtained the exact conditional distributions of the maximum likelihood estimators of the scale parameters. Since the conditional distributions of the maximum likelihood estimators of the scale parameters depend on the unknown location parameter, it is not possible to obtain the exact confidence intervals of the scale parameters based on the exact conditional distributions. We have proposed to use the Fisher information matrix to construct the asymptotic confidence intervals of the unknown scale parameters, assuming the location parameter to be known. We have also proposed to use the parametric bootstrap method for constructing confidence interval for the scale parameters, and it is very easy to implement it in practice. Extensive simulations have been performed to compare the performances of the different methods. One data analysis has been performed for illustrative purposes.

Analysis of simple step-stress model has also been performed when lifetimes have other distributions such as Weibull distribution, log-normal distribution, and generalized exponential distribution. Properties of the cumulative exposure model under Weibull distribution was studied in Komori [86]. Inferential aspects of step-stress model under Type-I and Type-II censoring schemes were addressed by Bai and Kim [6] and Kateri and Balakrishnan [83], respectively, when the distribution of lifetimes is assumed to be Weibull. In all these cases it was assumed that the model satisfies cumulative exposure model assumptions. However, it is noticed that maximum likelihood estimators of the unknown parameters do not exist in close form and therefore finding maximum likelihood estimators of the unknown parameters involves extensive computation. Most of the further statistical analysis mainly relies on asymptotic distribution of the maximum likelihood estimators. Moreover, extension of the analysis provided in Bai and Kim [6] and Kateri and Balakrishnan [83] are not immediate for more general censoring situations. It seems that Bayesian analysis is a natural choice in this case also. It may be worth mentioning that though some inferential issues on the parameters of Weibull distribution under the step-stress model have been addressed, no attention has been paid to develop the inference imposing the order restriction on the mean lifetime of the experimental units at different stress levels. Again a frequentist approach to the order restricted inference for parameters of Weibull distribution under step-stress model will be quite involved and hence, in this case also Bayesian approach is a natural alternative. In Chapter 6, we consider a simple step-stress model when the lifetimes are assumed to have two-parameter Weibull distribution. The analysis has been performed under the assumption that the model satisfies Khamis-Higgins model (see Khamis and Higgins [85]) assumptions. We have assumed quite flexible priors on the unknown parameters. It has been noticed that the Bayes estimators do not exist in close form in most of the cases. Therefore, an importance sampling based procedure has been proposed to calculate Bayes estimate and to construct Bayesian credible interval in

both the ordered restricted and unrestricted cases. Extensive simulations have been performed to examine performance of the proposed methods. Analysis of a data set has been also performed for illustrative purpose.

To

the fond memories of

Chotto-amma and Dadu

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Ayon Ganguly
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Nomenclature

i.i.d.	Identically and independently distributed.
AE	Average estimate.
AL	Average length.
ALT	Accelerated life test.
BE	Bayes estimator/estimate.
CDF	Cumulative distribution function.
CEM	Cumulative exposure model.
CI	Confidence interval.
CP	Coverage Percentage.
CRI	Credible interval.
CS-I	Type-I censoring scheme.
CS-II	Type-II censoring scheme.
EM	Expectation maximization.
GHCS-I	Generalized hybrid Type-I .
GHCS-II	Generalized hybrid Type-II .
HCS-I	Hybrid Type-I censoring scheme.
HCS-II	Hybrid Type-II censoring scheme.
HPD	Height posterior density.
HRF	Hazard rate function.
KHM	Khamis-Higgins model.
MGF	Moment generating function.
MLE	Maximum likelihood estimator/estimate.
MSE	Mean squared error.
PCS-I	Progressive Type-I censoring scheme.
PCS-II	Progressive Type-II censoring scheme.
PHCS-II	Progressive hybrid Type-II censoring scheme.
PMF	probability mass function.
SSLT	Step stress life test.
TFRM	Tampered failure rate model.
TRVM	Tampered random variable model.

Chapter 1

Introduction

1.1 Life Testing Experiments and its Difficulties

Life testing experiments have gained popularity in recent times. The main aim of a life testing experiment is to measure one or more reliability characteristics of the product under consideration. In a very classical form of life testing experiment, certain number of identical items are placed on the test under normal operating conditions and the 'time to failure' of all the items are recorded. The definition of 'time to failure' depends on the item considered. For examples, 'time to failure' may be the time after which a minimum satisfactory performance is not achieved for an electronic equipment, or it may be the number of revolution before malfunctioning for a ball bearing. For testing the lifetime of a electric bulb 'time to failure' is the hours it works before it is fused. The failure may occur due to any one or combination of more than one of the following reasons: (a) careless planning, (b) substandard raw materials, (c) random cause, (d) wear-out or fatigue caused by the aging of the item. As the time to failure can occur at any time, it is supposed that the time to failure is a random variable having a CDF.

However, due to the substantial improvement of the science and technology, the most of the products nowadays are quite durable and hence, one of the major difficulties of the life testing experiments is the time duration of the experiment. Also the most of the life testing experiments are destructive in nature, *i.e.*, items put on the test cannot be used for future purpose. To overcome these problems, experimenters use alternative techniques. Among them, censoring and ALT are playing important roles.

Censoring simply means truncation of the experiment before all the items put on the test fail. Depending the truncation criteria there exists different types of censoring schemes. In an ALT items are put on the test under some extreme operational conditions which affect the lifetime of the item under consideration negatively, *i.e.*, items are failed more quickly than the normal conditions. The factors which affect the lifetime of an item are called the stress factors. For example, voltage, temperature, humidity could be stress factors for an electronic equipment. ALT enables the experimenter to get more failures within a shorter time period and hence cut down the experimental time. SSLT, which provides freedom of changing the stress level during the experiment, is a special type of ALT. In this dissertation we will consider either one or the combination of both censoring and SSLT techniques. We will briefly discuss different censoring schemes and SSLT in the next two sections.

1.2 Censoring Schemes

Censoring is a very useful technique in life testing experiments. This is a technique to truncate the experiment in a well planned manner before the failure of all the items put on the test. Censoring can be done with respect to a pre-specified time or pre-specified number of failures or a combination of both. Depending upon the censoring criteria there are different types of censoring schemes. Consider the following experiment. Let n be a positive integer. A total of n items are put on the life testing experiment, and the time to failure of the items are recorded in a chronological order. Let $t_{1:n} < t_{2:n} < \ldots < t_{n:n}$ be the ordered failure times of the items. In all the cases it is assumed that the failed items are not replaced. In the next three subsections we will discuss some of the popular censoring schemes.

1.2.1 Basic Censoring Schemes

There are two very basic censoring schemes, *viz.*, CS-I and CS-II. They are the most common and popular censoring schemes.

Type-I Censoring Scheme

Let τ be a prefixed time. In a CS-I the experiment is terminated at the time τ. Hence, under this censoring scheme data set is of any one of the following forms:

(a) $t_{1:n} < t_{2:n} < \ldots < t_{N:n} < \tau$,

(b) there is no failure before the time τ,

where $N \in \{1, 2, \ldots, n\}$ is the number of failures before the time τ. Note that experimental time is fixed under this censoring scheme, but number of failures varies from experiment to experiment. Clearly, pre-fixed experimental duration is the advantage of this censoring scheme. However, wrongly chosen τ may end up in a few or even no failure before the time τ. If there were few failures, efficiency of further statistical analysis will be quite poor. Though statistical inference is possible in case of no failure, the results may not be informative, see Meeker and Escobar [107] and Nelson [114]. This is a major drawback of CS-I censoring scheme. Interested readers are referred to Lawless [93], Miller [110], and Bain and Englehardt [8] in this respect.

Type-II Censoring Scheme

Let $r(\leq n)$ be a prefixed positive integer. In a CS-II the experiment is terminated as soon as the r-th failure occurs. Under a CS-II the data set looks like

(a) $t_{1:n} < t_{2:n} < \ldots < t_{r:n}$.

On contrast to the previous censoring scheme, in this case number of failures is prefixed, but the experimental time varies from experiment to experiment. Clearly, pre-fixed number of failures is the main advantage, whereas no upper bound of the experimental duration is the main disadvantage of CS-II. Interested readers are referred to Lawless [93], Miller [110], and Bain and Englehardt [8].

1.2.2 Hybrid and Generalized Hybrid Censoring Schemes

HCSs are mixture of these two basic censoring schemes. Here we discuss the experimental setup of the HCS-I, HCS-II, GHCS-I, and GHCS-II.

Hybrid Type-I Censoring Scheme

Epstein [69] first introduced HCS-I and this censoring scheme can be described as follows. Let $r(\leq n)$ be a pre-chosen positive integer, and τ be a pre-determined time. The test is terminated when r-th item fails or time τ is reached on the test, whichever is earlier, *i.e.*, the termination time of the experiment is $\tau^* = \min\{t_{r:n}, \tau\}$. For HCS-I, the available data will be of the form

(a) $t_{1:n} < t_{2:n} < \ldots < t_{r:n}$ if $\tau^* = t_{r:n}$,
(b) $t_{1:n} < t_{2:n} < \ldots < t_{N:n}$ if $\tau^* = \tau$,
(c) there is no failure before the time τ,

where $N \in \{1, \ldots, r-1\}$ is the number of failures before the time τ. Note that the maximum duration of the experiment under this censoring scheme is τ and this is the main advantage of this censoring scheme. Like the CS-I, the experiment can be terminated with few or no failure before the time τ, and this is a serious disadvantage of HCS-I.

Hybrid Type-II Censoring Scheme

To overcome the disadvantage of the HCS-I by ensuring a minimum number of failures, Childs et al. [51] proposed HCS-II. Let $r(\leq n)$ be a pre-chosen positive integer, and τ be a pre-determined time. The test is terminated when r-th item fails or time τ is reached on the test, whichever is later, *i.e.*, the termination time of the experiment is $\tau^* = \max\{t_{r:n},\, \tau\}$. For HCS-II, the available data will be of the form

(a) $t_{1:n} < t_{2:n} < \ldots < t_{r:n}$ if $\tau^* = t_{r:n}$,

(b) $t_{1:n} < t_{2:n} < \ldots < t_{N:n}$ if $\tau^* = \tau$,

where $N \in \{0,\, 1,\, \ldots,\, n\}$ is the number of failures before the time τ. Note that in the second case N is restricted to the set $\{r,\, r+1,\, \ldots,\, n\}$. However, this censoring scheme has no upper bound on the time duration which is the main disadvantage of HCS-II.

Generalized Hybrid Type-I Censoring Scheme

To overcome the drawback of HCS-I and HCS-II, Chandrasekar et al. [45] proposed GHCS-I and GHCS-II. Let r and k be two prefixed positive integers satisfying $k < r \leq n$ and $\tau \in (0, \infty)$ be a predetermined time. If the k-th failures occurs before the time τ, the experiment is terminated at $\min\{t_{r:n},\, \tau\}$. If the k-th failure occurs after the time τ, the experiment is terminated at the time $t_{k:n}$. Note that GHCS-I modifies HCS-I by allowing the experiment to continue beyond the time τ. Under this censoring scheme the experimenter would like to observe r failures but is willing to accept a bare minimum of k failures.

Generalized Hybrid Type-II Censoring Scheme

This censoring scheme was also introduced by Chandrasekar et al. [45]. Let $r\,(\leq n)$ be a prefixed positive integer and $\tau_1,\, \tau_2 \in (0, \infty)$ be two pre-specified times such that

$\tau_1 < \tau_2$. If the r-th failure occurs before the time τ_1, the experiment is terminated at the time τ_1. If the r-the failure occurs between the times τ_1 and τ_2, the experiment stops at the time $t_{r:n}$. Otherwise the experiment is terminated at the time τ_2. Thus GHCS-II modifies the HCS-II by reducing the experimental time by τ_2 from above. Interested readers are referred to the review article by Balakrishnan and Kundu [28] in this regard.

1.2.3 Progressive Censoring Schemes

The PCSs enable the experimenter to remove unit at working condition from the test before the termination of the experiment. This type of working unit can be used in further testing.

Progressive Type-I Censoring Scheme

Let $k\,(\leq n)$ be an pre-specified positive integer. Let $\tau_1 < \tau_2 < \ldots < \tau_k$ be pre-determined k time points, and $R_1, \ldots, R_{k-1}$ be pre-specified $(k-1)$ non-negative integers. Let N_1 be the number of failures before the time τ_1. R_1 items are randomly chosen from the remaining $(n - N_1)$ units and removed from the test at the time point τ_1. The experiment continues and suppose N_2 is the number of failures between the times τ_1 and τ_2. Out of $(n - N_1 - R_1 - N_2)$ units still on the experiment, randomly chosen R_2 units are removed form the test at the time point τ_2, and so on. Finally at the time point τ_k all the remaining items, say R_k, are censored and the experiment is terminated. Note that a PCS-I is feasible if the number of units still on the test at each censoring time is larger than the number of items planned to censor at that time point, and feasibility of such a censoring scheme is always assumed, see Balakrishnan [9]. Clearly, the experiment termination time is fixed at τ_k, and we have the relation $\sum_{j=1}^{k}(N_j + R_j) = n$.

Progressive Type-II Censoring Scheme

Let $k\,(\leq n)$ be an pre-specified positive integer. Let R_1, R_2, $\ldots$, R_k be k prefixed non-negative integers satisfying $k + \sum_{j=1}^{k} R_j = n$. At the point of the first failure, $t_{1:n}$, randomly selected R_1 items are removed form the remaining $(n-1)$ units. Similarly, randomly selected R_2 units out of $(n-2-R_1)$ remaining items are removed at the point of second failure, $t_{2:n}$, and so on. Finally, at the point of k-th failure, $t_{k:n}$, remaining R_k units are censored form the test, and the experiment terminates.

Progressive Hybrid Type-II Censoring Scheme

Let $k\,(\leq n)$ be an pre-specified positive integer, and R_1, R_2, $\ldots$, R_k be k prefixed non-negative integers such that $k + \sum_{j=1}^{k} R_j = n$. Let τ be a predetermined time. At the time of first failure, $t_{1:n}$, randomly selected R_1 items are removed form the remaining $(n-1)$ items. At the time of the second failure, $t_{2:n}$, randomly selected R_2 items out of remaining $(n-2-R_1)$ items are censored, and so on. If the k-th failure occurs before the time point τ, remaining R_k items are removed from the test at the time $t_{k:n}$, and the experiment terminates. On the other hand if there are fewer failures than k before the time τ, the experiment is terminated at the time point τ by removing all the remaining items form the test. Clearly, in the second case the number of items censored at the last stage is $R_N^* = n - N - \sum_{j=1}^{k} R_j$, where $0 \leq N < k$ is the number of failures before the time τ. Hence, under the PHCS-II the observed data will be of the form

(a) $t_{1:n} < \ldots < t_{k:n}$ if $t_{k:n} < \tau$,

(b) $t_{1:n} < \ldots < t_{N:n}$ if $t_{N:n} < \tau < t_{N+1:n}$.

For more detailed description of PHCS-II, readers are referred to Kundu and Joarder [92].

1.3 Accelerated Life Tests

In many life testing experiments it is very difficult to observe sufficient number of failures in an affordable time under the normal operating conditions. This is due to the rapid increase in reliability of the products to cope with the competition. ALTs are introduced to overcome this problem by allowing the experimenter to conduct the experiment under one or more extreme operating conditions and thus increasing the number of failures within an affordable experimental time. The factors which directly effect the lifetime of the product under consideration are called stress factors, *e.g.*, voltage, temperature, humidity could be some of the stress factors for testing an electronic equipment.

A special case of ALT is SSLT, which enables the experimenter to change the levels of a stress factor in a sequential manner during the experiment. Let $s_1, s_2, \ldots, s_k$ be k predetermined stress levels and $\tau_1 < \tau_2 < \ldots < \tau_{k-1}$ be $(k-1)$ pre-specified time points. In a very basic form of SSLT, n units are put on the test at an initial stress level s_1. At the time point τ_1, the stress level is changed to s_2 form s_1. Similarly the stress level is changed to s_3 form s_2 at the time point τ_2 and so on. Finally at the time point τ_{k-1}, the stress level is changed form s_{k-1} to s_k. Experiment stops when all the items put on the test fail. The failure times are recorded in a chronological order. If we assume that the number of failures before the time τ_i, $i = 1, 2, \ldots, k-1$, is n_i satisfying $n_1 \leq n_2 \leq \ldots \leq n_{k-1}$, then a typical data set looks like $t_{1:n} < \ldots < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < \ldots < t_{n_2:n} < \tau_2 < \ldots < \tau_{k-1} < t_{n_{k-1}+1} < \ldots < t_{n:n}$. A SSLT is called simple SSLT if $k = 2$, *i.e.*, there are only two stress levels. However, in most of the situations a SSLT is performed in presence of a censoring scheme and hence, we briefly present the form of observed data in the next subsection.

1.3.1 Form of Data under Different Censoring Schemes

A total of n units is placed on a simple SSLT experiment. The stress level is changed from s_1 to s_2 at a prefixed time τ_1, and $\tau_2 > \tau_1$ is another prefixed time. The positive integer $r(\leq n)$ is also pre-fixed. The role of r and τ_2 will be clear later. Let the ordered lifetimes of these items be denoted by $t_{1:n} < \ldots < t_{n:n}$. Let N_1 and N_2 denote the number of failures before the time τ_1 and between time τ_1 and τ_2, respectively.

Type-I Censoring Scheme

The test is terminated when the time τ_2 on the test has been reached. For Type-I censoring the available data is of the form

(a) $\tau_1 < t_{1:n} < \ldots < t_{N_2:n} < \tau_2$,
(b) $t_{1:n} < \ldots < t_{N_1:n} < \tau_1 < t_{N_1+1:n} < \ldots < t_{N_1+N_2:n} < \tau_2$,
(c) $t_{1:n} < \ldots < t_{N_1:n} < \tau_1 < \tau_2$.

Type-II Censoring Scheme

The test is terminated when the r-th failure takes place, *i.e.*, it is terminated at a random time $t_{r:n}$. In this case the available data is of the form

(a) $\tau_1 < t_{1:n} < \ldots < t_{r:n}$,
(b) $t_{1:n} < \ldots < t_{N_1:n} < \tau_1 < t_{N_1+1:n} < \ldots < t_{r:n}$; $N_1 < r$,
(c) $t_{1:n} < \ldots < t_{r:n} < \tau_1 < \tau_2$.

Type-I Hybrid Censoring Scheme

In this case, the test is terminated at a random time $\tau^* = \min\{t_{r:n}, \tau_2\}$. For HCS-I, the available data is of the form

(a) $\tau_1 < t_{1:n} < \ldots < t_{r:n}$ if $t_{r:n} < \tau_2$,

(b) $t_{1:n} < \ldots < t_{N_1:n} < \tau_1 < t_{N_1+1:n} < \ldots < t_{r:n}$ if $t_{r:n} < \tau_2$, $N_1 < r$,

(c) $t_{1:n} < \ldots < t_{r:n} < \tau_1$ if $t_{r:n} < \tau_2$,

(d) $\tau_1 < t_{1:n} < \ldots < t_{N_2:n} < \tau_2$ if $t_{r:n} > \tau_2$,

(e) $t_{1:n} < \ldots < t_{N_1:n} < \tau_1 < t_{N_1+1:n} < \ldots < t_{N_1+N_2:n} < \tau_2$ if $t_{r:n} > \tau_2$, $N_1 < r$,

(f) $t_{1:n} < \ldots < t_{N_1:n} < \tau_1 < \tau_2$ if $t_{r:n} > \tau_2$.

Type-II Hybrid Censoring Scheme

In HCS-II, the experiment is terminated at a random time $\tau^* = \max\{t_{r:n}, \tau_2\}$. In this case the available data is of the form

(a) $\tau_1 < t_{1:n} < \ldots < t_{r:n}$ if $t_{r:n} \geq \tau_2$,

(b) $t_{1:n} < \ldots < t_{N_1:n} < \tau_1 < t_{N_1+1:n} < \ldots < t_{r:n}$ if $t_{r:n} \geq \tau_2$, $N_1 < r$,

(c) $\tau_1 < t_{1:n} < \ldots < t_{N_2:n} < \tau_2$ if $t_{r:n} < \tau_2$,

(d) $t_{1:n} < \ldots < t_{N_1:n} < \tau_1 < t_{N_1+1:n} < \ldots < t_{N_1+n_2:n} < \tau_2$ if $t_{r:n} < \tau_2$,

(e) $t_{1:n} < \ldots < t_{N_1:n} < \tau_1 < \tau_2$ if $t_{r:n} < \tau_2$.

Type-II Progressive Censoring Scheme

$R_1, \ldots, R_m$ are m prefixed non-negative integers such that

$$m + \sum_{j=1}^{m} R_j = n.$$

At the time of the first failure, say $t_{1:n}$, R_1 units are chosen at random from the remaining $(n-1)$ units and they are removed from the experiment. Similarly, at the time of the second failure, say $t_{2:n}$, R_2 units are chosen at random from the remaining $(n - R_1 - 2)$ surviving units and they are removed from the test, and so on. Finally at the time of the mth failure, say $t_{m:n}$, the rest of the $R_m = n - m - \sum_{j=1}^{m-1} R_j$ units are removed and the experiment is stopped. In this case the available data is of the form

(a) $\tau_1 < t_{1:n} < \ldots < t_{m:n}$ if $\tau_1 < t_{1:n}$,

(b) $t_{1:n} < \ldots < t_{N_1:n} < \tau_1 < t_{N_1+1:n} < \ldots < t_{m:n}$ if $t_{1:n} < \tau_1 < t_{m:n}$,

(c) $t_{1:n} < \ldots < t_{m:n} < \tau_1$ if $\tau_1 \geq t_{m:n}$.

1.3.2 Existing Models

Let us assume that the CDF of the lifetime at the stress level s_i is $F_i(\cdot)$. To analyze a data observed under a SSLT, one needs a model which relates the CDFs of lifetime under different stress levels to the CDF of lifetime of the product under the SSLT. Several models have been proposed in the literature to describe this type of relationship. Among them the most popular and commonly used one is CEM, first proposed by Seydyakin [126] and later quite extensively studied by Nelson [112]. This model assumes that the remaining lifetime of an unit depends only on the cumulative exposure accumulated and current stress level, regardless of how the exposure is actually accumulated. Moreover, at a fixed stress level unit will fail according to the CDF of that stress level only starting at previous fraction accumulated. To construct the CEM, which ensures the continuity of the resultant CDF, let us consider a k step SSLT, where stress levels are changed at prefixed time $\tau_1 < \tau_2 < \ldots < \tau_{k-1}$. Clearly, the units under test will fail according to the CDF of the stress level s_1 till the time τ_1 and hence,

$$F(t) = F_1(t) \quad \text{for} \quad 0 \leq t < \tau_1.$$

The effect of change of the stress level from s_1 to s_2 at the time point τ_1 is equivalent to change the CDF of stress level s_2 from $F_2(t)$ to $F_2(t - h_1)$, *i.e.*,

$$F(t) = F_2(t - h_1) \quad \text{for} \quad \tau_1 \leq t < \tau_2,$$

where the shifting parameter h_1 is the solution of the equation

$$F_2(\tau_1 - h_1) = F_1(\tau_1).$$

Proceeding in this way, finally one will have, with $\tau_0 = 0$ and $\tau_k = \infty$

$$F(t) = F_i(t - h_{i-1}) \quad \text{for} \quad \tau_{i-1} \leq t < \tau_i, \quad i = 1, 2, \ldots, k, \tag{1.1}$$

where $h_0 = 0$ and h_i, $i = 1, 2, \ldots, k-1$, is the solution of the equation

$$F_{i+1}(\tau_i - h_i) = F_i(\tau_i - h_{i-1}).$$

Let us consider an example of CEM. Here we assume $k = 3$ and the lifetime of the units under consideration is exponentially distributed with scale parameter (mean)

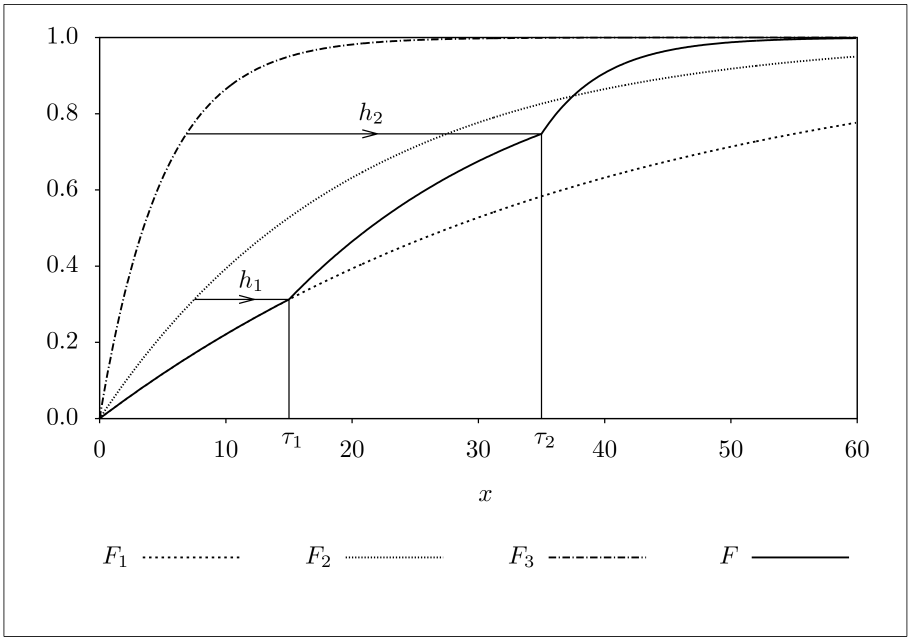


Figure 1.1: Pictorial presentation of CDF under CEM.

θ_i at the stress level s_i, $i = 1, 2, 3$, *i.e.*, the CDF of the lifetime at the stress level s_i is given by

$$F_i(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-\frac{t}{\theta_i}} & \text{if } t \geq 0. \end{cases}$$

The CDF of the lifetime under the assumption of CEM can be obtained as follows. Till the time τ_1, it is identically same as the CDF of lifetime at the stress level s_1. For $t \in (\tau_1, \tau_2)$, $F(t) = F_2(t - h_1)$, where h_1 is the solution of the equation

$$1 - e^{-\frac{\tau_1}{\theta_1}} = 1 - e^{-\frac{\tau_1 - h_1}{\theta_2}},$$

which implies $h_1 = (1 - \theta_2/\theta_1)\tau_1$. Similarly, for $t \in (\tau_2, \infty)$, the CDF of lifetime under the SSLT is given by $F(t) = F_3(t - h_2)$, where $h_2 = (1 - \theta_3/\theta_2)\tau_2 + (1/\theta_2 - 1/\theta_1)\theta_3\tau_1$. Hence, finally we have

$$F(t) = \begin{cases} 1 - e^{-\frac{t}{\theta_1}} & \text{if } \ 0 \leq t < \tau_1 \\ 1 - e^{-\frac{t-\tau_1}{\theta_2} - \frac{\tau_1}{\theta_1}} & \text{if } \ \tau_1 \leq t < \tau_2 \\ 1 - e^{-\frac{t-\tau_2}{\theta_3} - \frac{\tau_2 - \tau_1}{\theta_2} - \frac{\tau_1}{\theta_1}} & \text{if } \ t \geq \tau_2. \end{cases} \tag{1.2}$$

The process of obtaining CDF under the assumption of CEM is depicted in the Figure 1.1, where we take $\theta_1 = 40$, $\theta_2 = 20$, $\theta_3 = 5$, $\tau_1 = 15$, and $\tau_2 = 35$.

The TRVM was proposed by DeGroot and Goel [61] for a simple SSLT and assumes that the effect of the change of the stress level from s_1 to s_2 at the time τ_1 is equivalent to multiply the remaining life of the unit by an unknown positive constant, say α, which depends on both the stress levels. Mathematically, if T denote the lifetime under the stress level s_1, then the lifetime, $\widetilde{T}$, under the simple SSLT is given by

$$\widetilde{T} = \begin{cases} T & \text{if } T \leq \tau_1 \\ \tau_1 + \alpha(T - \tau_1) & \text{if } T > \tau_1. \end{cases} \tag{1.3}$$

Note that for a CEM the CDF of the lifetime under different stress level may be fully unrelated. However, for the TRVM, they are related to each other by (1.3). DeGroot and Goel [61] considered the optimal design in the Bayesian framework under the assumption that the distribution of T is exponential, and the stress changing time is different for different experimental units.

The TFRM was first proposed by Bhattacharyya and Soejoeti [42] for simple SSLT and then generalized by Madi [102] for the multiple step SSLT. Bhattacharyya and Soejoeti [42] assumed that the effect of switching the stress level form s_1 to s_2 is to multiply the failure rate of the stress level s_1 by an unknown constant α, *i.e.*,

$$\widetilde{\lambda}(t) = \begin{cases} \lambda(t) & \text{if } t \leq \tau_1 \\ \alpha\lambda(t) & \text{if } t > \tau_1, \end{cases} \tag{1.4}$$

where $\lambda(\cdot)$ and $\widetilde{\lambda}(\cdot)$ are the failure rate at the stress level s_1 and that under simple SSLT. It is worth mentioning here that the TRVM and TFRM will coincide if $\alpha = 1$ or the lifetime distribution under the stress level has lack of memory property, see Rao [124]. Madi [102] generalized the concept of Bhattacharyya and Soejoeti [42], incorporating multi-step SSLT. They assumed that the failure rate, $\widetilde{\lambda}(t)$, under SSLT is given by

$$\widetilde{\lambda}(t) = \left(\prod_{j=0}^{i-1} \alpha_j\right) \lambda(t) \quad \text{if } \tau_{i-1} \leq t < \tau_i \quad i = 1, 2, \ldots, k,$$

where $\tau_0 = 0$, $\tau_k = \infty$, $\alpha_0 = 1$ and $\alpha_i > 0$ for $i = 1, 2, \ldots, k-1$. If we take $\lambda(t) = \beta x^{\beta-1}/\theta$, *i.e.*, the lifetime under the stress level s_1 has a Weibull distribution with shape parameter β and scale parameter θ, the CDF of the lifetime under the step-stress pattern is given by

$$F(t) = 1 - e^{-\left(t^{\beta} - \tau_{i-1}^{\beta}\right)/\theta_i - \sum_{j=1}^{i-1}\left(\tau_j^{\beta} - \tau_{j-1}^{\beta}\right)/\theta_j} \quad \text{if } \tau_{i-1} < t \leq \tau_i, \quad i = 1, 2, \ldots, k, \tag{1.5}$$

where $\theta_1 = \theta$ and $\theta_i = \theta / \left(\prod_{j=1}^{i-1} \alpha_j\right)$ for $i = 2, 3, \ldots, k$. This model was also proposed by Khamis and Higgins [85] when lifetime at the stress level s_i has a Weibull distribution with common shape parameter β and different scale parameter θ_i and was named as KHM. Xu and Tang [140] showed that the KHM is actually a special case of TFRM. Note that the CDF of Weibull distribution coincides with the CDF of exponential distribution under power transformation, which is extensively used to prove many properties of Weibull distribution. However, the CDF of CEM for Weibull distribution does not coincide with CDF of CEM for the exponential distribution under power transformation, whereas CDF under KHM assumptions coincides with CDF of CEM for exponential distribution under the same transformation. This is main advantage of the KHM over the CEM.

1.4 Distributions Used in this Dissertation

Several distributions have been used in the literature to describe the life pattern of different items. Some popular distributions in the domain of reliability are exponential, Weibull, gamma, log-normal, and extreme value distributions. We have mainly used exponential and Weibull distributions in this dissertation as the lifetime distribution. We have also used gamma distribution in different contexts . Hence, in this section we provide a brief discussion on these three distributions.

Exponential Distribution

One-parameter exponential distribution is characterized by PDF

$$f(t) = \lambda\, e^{-\lambda t} \quad \text{for} \quad t > 0,$$

where $\lambda > 0$. The corresponding CDF and HRF are given by

$$F(t) = 1 - e^{-\lambda t} \quad \text{for} \quad t > 0, \qquad\qquad h(t) = \lambda \quad \text{for} \quad t > 0,$$

respectively. The mean and variance of this distribution function are $\dfrac{1}{\lambda}$ and $\dfrac{1}{\lambda^2}$, respectively, while p-th (for $0 < p < 1$) percentile point is $-\dfrac{\ln(1-p)}{\lambda}$. The distribution where $\lambda = 1$ is called standard exponential distribution. This distribution is the first widely used lifetime distribution. The reasons of its popularity are simple representation of its PDF, CDF, and HRF, availability of simple statistical methods for data analysis, and ability to adequately fit the lifetime of several types of manufactured items. However, the constant hazard rate and sensitivity of inferential procedures when original model departs from the exponential model lead to caution in the use of this distribution. It may be mentioned here that this distribution has lack of memory property, which implies that for $t_1 > 0$, $t_2 > 0$

$$P(X > t_1 + t_2 \,|X > t_1) = P(X > t_2),$$

if the random variable X has a exponential distribution.

Two-parameter exponential distribution is characterized by PDF

$$f(t) = \lambda\, e^{-\lambda(t-\mu)} \quad \text{for} \quad t > \mu,$$

where $\infty < \mu < \infty$ and $\lambda > 0$. In this case μ is called the location parameter, whereas λ is known as the scale parameter. Note that if X has a two-parameter exponential distribution, $X - \mu$ is distributed as a one-parameter exponential distribution.The mean and variance of this distribution function are $\mu + \dfrac{1}{\lambda}$ and $\dfrac{1}{\lambda^2}$, respectively, while p-th (for $0 < p < 1$) percentile point is $\mu - \dfrac{\ln(1-p)}{\lambda}$.

Weibull Distribution

Weibull distribution, perhaps the most widely used lifetime distribution, is characterized by the PDF

$$f(t) = \beta\lambda\, t^{\beta-1}\, e^{-\lambda t^{\beta}} \quad \text{for} \quad t > 0$$

with $\beta > 0$ and $\lambda > 0$. The corresponding CDF and HRF are given by

$$F(t) = 1 - e^{-\lambda t^{\beta}} \quad \text{for} \quad t > 0, \qquad\qquad h(t) = \beta\lambda t^{\beta-1} \quad \text{for} \quad t > 0,$$

respectively. The shape and spread of the PDF depend on the parameter β and λ, respectively, for which they are called shape and scale parameters, respectively. Note that if $\beta < 1$, HRF is a decreasing function in t, whereas HRF is an increasing function in t provided $\beta > 1$. For $\beta = 1$, HRF is constant and it corresponds to the exponential distribution. The mean and variance of this distribution are $\dfrac{\Gamma(1+1/\beta)}{\lambda^{1/\beta}}$ and $\dfrac{1}{\lambda^{1/\beta}}\left[\Gamma(1+2/\beta) - \Gamma(1+1/\beta)^2\right]$, respectively. It may be mentioned here that Weibull distribution can be positively or negatively skewed depending upon the value of the shape parameter β. If $\beta > \beta_0 = 3.6023494257197$, the distribution is negatively skewed, whereas it is positively skewed for $\beta < \beta_0$, see Cohen and Whitten [59]. The reason for the popularity of this distribution is its simple form of PDF, CDF and HRF. This distribution is very flexible to fit different types of lifetime data, and it can be used quite conveniently even for censored data also.

Gamma Distribution

The PDF of a two-parameter gamma distribution is given by

$$f(t) = \frac{\lambda^{\beta}}{\Gamma(\beta)}\, t^{\beta-1}\, e^{-\lambda t} \quad \text{for} \quad t > 0$$

with $\beta > 0$ and $\lambda > 0$. The close form of the CDF does not exist in this case, and as a result HRF is also not in close form. However, it can be shown that HRF is a decreasing function in t with $\lim_{t\to 0} h(t) = \infty$ and $\lim_{t\to\infty} h(t) = \lambda$ for $\beta < 1$, whereas it is an increasing function with $h(0) = 0$ and $\lim_{t\to\infty} h(t) = \lambda$ for $\beta > 1$. If the shape parameter is one, this distribution coincides with a one-parameter exponential distribution. Again for $\beta \leq 1$, gamma distribution has a J-shaped PDF, whereas its PDF is bell shaped for $\beta > 1$. As the CDF and HRF do not exist in close form, further analysis of this distribution can be complicated. Though gamma distribution fits different types of lifetime data, we never assume that the distribution of lifetime of the product under consideration has a gamma distribution in this dissertation. We encounter this distribution in the sampling distribution of the MLE of mean lifetime in Chapters 2, and 5 and as prior distribution in Chapters 3, 4 and 6.

1.5 Literature Review

1.5.1 Censoring Scheme

CS-I and CS-II are the most studied censoring schemes. In this regard some of the references are Lawless [93], Miller [110], Bain and Englehardt [8] in frequentist framework. Some Bayesian treatment can be found in Hamada et al. [78]. Epstein [69] has proposed the HCS-I. Different inferential issues of exponential model under HCS-I have been addressed by several authors. A set of two sided confidence interval for mean lifetime was proposed by Fairbanks et al. [70], where the data are exponentially distributed and are hybrid Type-I censored. Chen and Bhattacharya [47] derived the MLE of the mean lifetime and the distribution of the MLE under the same setup. However, an equivalent but easier form of the PDF of the MLE of the mean lifetime was derived by Childs et al. [51]. Chen and Bhattacharya [47] and Childs et al. [51] used the CDF of the MLE of the mean lifetime to find the exact CI based on the assumption that the tail probability $P_\theta(\widehat{\theta} > b)$ is an increasing function of θ for all fixed b. However, the authors did not provide any formal proof of this assumption, which has been formally proved by Balakrishnan and Iliopoulos [19] recently. Draper and Guttman [62] considered estimation of parameter of exponential distribution under the Bayesian framework when data are hybrid Type-I censored. Gupta and Kundu [77] compared different methods of estimation for one-parameter exponential distribution using an extensive Monte Carlo simulation. Ebrahimi [67] considered the maximum likelihood estimation of model parameters when the data are assumed to follow a two-parameter exponential distribution, and both with and without replacement cases were considered in this article. However, exact distribution of the MLE was not derived in Ebrahimi [67]. The exact distribution of the MLEs of the location and scale parameters were derived by Childs et al. [49]. Though the exponential distribution is studied more extensively in the literature, other distributions are not left out. Kundu [87] considered the Weibull distribution under the HCS-I. Both the maximum likelihood estimation and the Bayesian estimation are considered in this article. Dube et al. [65] developed the maximum likelihood estimation of the model parameters of log-normal distribution under the assumption of HCS-I. They have suggested the use of EM algorithm for computation of MLEs in this case. HCS-I has been extensively used in the context of reliability acceptance in MIL-STD-781C [108].

HCS-II was introduced by Childs et al. [51]. They considered the frequentist estimation of the mean lifetime under the assumption of one-parameter exponentially distributed lifetime. Exact PDF of the MLE was derived in the same article. Like HCS-I, Childs et al. [51] used the exact CDF to construct the CI for the mean lifetime based on the assumption that $P_\theta(\widehat{\theta} > b)$ is an increasing function of the mean lifetime θ for all fixed b. However, Childs et al. [51] did not provide any formal proof of this assertion and the assumption was later proved by Balakrishnan and Iliopoulos [19]. Park et al. [122] derived the Fisher information for HCS-II. Banerjee and Kundu [38] proved the existence and uniqueness of the MLEs of the shape and scale parameters of the Weibull distribution when the data are hybrid Type-II censored. Banerjee and Kundu [38] suggest to solve the profile likelihood function of the shape parameter using fixed point iteration method under the same setup. Analysis of the hybrid Type-II censored data having a two-parameter Weibull distribution under the Bayesian framework was also presented in Banerjee and Kundu [38]. Extensive Monte Carlo simulation was done by Banerjee and Kundu [38] to compare the performance of the different estimation procedures under the same set of assumptions.

GHCS-I and GHCS-II were introduced by Chandrasekar et al. [45]. The authors had also considered the estimation of the model parameter when the data are assumed to be distributed according to a exponential distribution. Assuming stochastic monotonicity of $\widehat{\theta}$ with respect to θ, Chandrasekar et al. [45] used the exact CDF to construct CI for both the generalized hybrid censoring schemes. However, the formal proof of the stochastic monotonicity was given by Balakrishnan and Iliopoulos [19] later. Fisher information for both the generalized hybrid censoring schemes were discussed by Park and Balakrishnan [120]. An extensive review of different hybrid censoring schemes can be found in a recent discussion article by Balakrishnan and Kundu [28].

Progressive censoring schemes were first discussed by Herd [80], who named it as 'multi-censoring'. Cohen [53] pointed out the usefulness of the progressive censoring schemes in reliability testing. For a book length account on progressive censoring, readers are referred to Balakrishnan and Aggarwala [11]. Though PCS-I is more natural and practical than PCS-II, former poses more difficulty to analyze the mathematical properties than the latter. Cohen [53] considered likelihood inference of the normal distribution under PCS-I. The Author showed that explicit solutions of likelihood equations do not exist, and one needs to use some numerical methods to solve them. They suggested the use of probit to find initial guesses. Gajjar and Khatri [73] considered the PCS-I under log-normal and logistic models when population parameters change after each censoring time point. The inference of the unknown parameters of two-parameter Weibull and three-parameter Weibull distributions under PCS-I were discussed by Cohen [54] and Cohen [55], respectively. For three-parameter Weibull distribution, Cohen [55] showed that if the value of the shape parameter of the Weibull distribution is less than one, MLEs do not exist, and he proposed to use modified MLEs to overcome this problem. Maximum likelihood estimation of three-parameter gamma distribution was considered by Cohen and Norgaard [58] under PCS-I. Maximum likelihood inference of three-parameter log-normal distribution under the same progressive censoring scheme was developed by Cohen [56]. Exact distribution of MLEs of parameters of one- and two-parameter exponential distributions have been considered by Cramer and Balakrishnan [60], when the data are Type-I progressively hybrid censored. The main tool used by the authors is the distribution of the spacings from Type-I progressively hybrid censored data. Least squares median rank estimator and MLEs of the two-parameter Weibull distribution were discussed by Gibbons and Vance [74]. Bayesian inference and life testing plan for Weibull distribution in presence of PCSs was considered by Kundu [88]. Wingo [133] considered the Burr Type-XII distribution under PCS-I, and proved the existence and uniqueness of solution to the likelihood equations. Development of PCS-I can be also found in Nelson [113], Cohen and Whitten [59], Cohen [57], and Balakrishnan and Cohen [15].

Under the assumption of existence of the PDF of the parent distribution, the joint density of general progressively Type-II censored order statistics is provided by Balakrishnan and Sandhu [33] and Aggarwala and Balakrishnan [1]. Thomas and Wilson [129] considered Weibull and extreme value distributions and Viveros and Balakrishnan [130] considered location-scale family of distribution under PCS-II. Thomas and Wilson [129] and Viveros and Balakrishnan [130] proved the independence of normalized spacing of progressively Type-II censored order statistic, when the data are coming from standard exponential distribution. The authors also showed that the normalized spaciness are identically distributed as standard exponential distribution in this case. A simple algorithm is proposed by Balakrishnan and Sandhu [32] for generating progressively Type-II censored data from continuous distributions. Several authors considered the likelihood inference of PCS-II under different distributional assumption, and here we provide a brief review of the same. Balakrishnan and Kannan [23] considered the MLEs of the unknown parameters for logistic distribution, when the data are progressively Type-II right censored. The likelihood estimation of the Laplace distribution was considered by Aggarwala and Balakrishnan [2] under the same censoring scheme. Balakrishnan et al. [24] and Balakrishnan et al. [25] considered MLEs of the unknown parameters of the gamma and extreme value distribution, respectively, whereas the likelihood inference of the scaled half-logistic distribution was discussed by Balakrishnan and Asgharzadeh [12] under the PCS-II. Zheng and Park [142] provided a decomposed form of Fisher information matrix under the PCS-II. The authors including Ng et al. [118], Ng et al. [119] and Lin et al. [98] used the EM algorithm under the PCS-II. Ng et al. [118] and Ng et al. [119] also considered the Fisher information using the technique of "missing information principle". Existence and uniqueness of the MLEs of the unknown parameters for the normal distribution was discussed by Balakrishnan and Mi [30], Balakrishnan and Kateri [26] considered the same for the Weibull distribution, when the data are progressively Type-II censored. Under PCS-II, Viveros and Balakrishnan [130] considered the interval estimation of unknown parameters and the functions of them for a density belongs to location-scale family of distributions using the conditional method proposed by Fisher [72]. Predictive intervals for the smallest life length from a future sample was also addressed by Viveros and Balakrishnan [130]. Similar approach for the interval estimation of the unknown parameters can be also found in Childs and Balakrishnan [48] for Laplace distribution and in Lin et al. [97] for log-gamma distribution under PCS-II. Exact predictive intervals for last censored failure time for exponential distribution was addressed by Balakrishnan and Lin [29] under PCS-II. Robinson [125] considered the bootstrap confidence interval for location and scale parameters of a PDF belonging to location-scale family of distributions under PCS-II. A restricted MLEs and likelihood test procedure for exponential distribution under PCS-II was developed by Bhattacharya [41]. The best linear unbiased estimators and other linear estimators along with their properties under variety of distributions were discussed by many authors including Mann [103], Mann [104], Thomas and Wilson [129], Cacciari and Montanari [44], Mantanari and Cacciari [105], Balakrishnan and Sandhu [33], Balakrishnan and Rao [31], Aggarwala and Balakrishnan [1], Balakrishnan and Aggarwala [11], Balakrishnan et al. [16], Balakrishnan et al. [14], Balakrishnan and Lin [29], Chandrasekar et al. [46], Burkschat et al. [43]. It is also a more general censoring mechanism than the traditional CS-I or CS-II, see for example the monograph by Balakrishnan and Aggarwala [11] and also the recent review article by Balakrishnan [9] in this respect.

Kundu and Joarder [92] considered the estimation of the parameter of one-parameter exponential distribution under both the classical and Bayesian framework when the data are progressively Type-I hybrid censored. They considered a gamma prior on mean lifetime and compared different methods of estimation using extensive simulation study. Exact distribution of the MLE of the mean lifetime of exponential distribution was developed by Childs et al. [50] under PHCS-I. However, the obvious drawback of this censoring scheme is the non-existence of the MLE under some circumstances. To overcome this drawback, Childs et al. [50] also introduced the PHCS-II and the exact distribution of MLE of the mean lifetime of exponentially failure data was considered in that article. Based on the assumption that $P_\theta(\widehat{\theta} > b)$ is a increasing function of θ for all fixed b, Childs et al. [50] developed exact CI for the mean lifetime for both PHCS-I and PHCS-II. However, the formal proof of this assumption remains a open problem till now. Park et al. [121] considered the Fisher information for the exponential failure data under both the progressive hybrid censoring schemes. Weibull distribution was studied by Mokhtari et al. [111]

under both the classical and Bayesian framework when the data are progressively hybrid Type-II censored.

1.5.2 Accelerated Life Tests

ALTs are gaining popularity in the recent times due to the increase in the life expectancy of several products and are being studied in the literature by several authors. Meeker and Escobar [107], and Bagdanavicius and Nikulin [4] provided some of the book length key references in this area. A nice overview and a nice bibliography of accelerated life test was provided by Nelson [116] and Nelson [117]. The CEM was first introduced by Seydyakin [126] and then discussed by Nelson [112] and Nelson [115]. DeGroot and Goel [61] proposed TRVM. In the same article optimality of a step-stress test was considered under the Bayesian setup. A TFRM, which was first proposed by Bhattacharyya and Soejoeti [42] for simple SSLTs, assumes that the failure rate of a stress level is same as that of the initial stress level multiplied by a suitable constant and was generalized by Madi [102] for multiple step SSLTs. SSLT is quite extensively studied in literature under the CEM and different censoring schemes. Balakrishnan et al. [27] considered point and interval estimation for a simple step-stress model with Type-II censoring when failure times are assumed to be exponentially distributed. Simple SSLT under CS-I was considered by Balakrishnan et al. [36] for exponential distribution. Balakrishnan et al. [27] and Balakrishnan et al. [36] constructed CI for the scale parameter based on the assumption that CDF of the MLE of the scale parameter is a decreasing function of that scale parameter when other quantities are held constant. However, they did not provide any formal proof. Balakrishnan and Iliopoulos [20] proved this assumption under both censoring schemes. Step-stress model under the presence of competing risks with exponentially distributed failure times is considered by Balakrishnan and Han [17] under SC-II. Exact inference for a exponential simple step-stress model with HCS-I and HCS-II were studied by Balakrishnan and Xie [34, 35], respectively. Gouno et al. [75] and Han et al. [79] considered step-stress model under PCS-I and optimality of the test. An analysis of simple step-stress models under the exponential CEM and PCS-II was considered by Xie et al. [135]. Gouno et al. [75] considered the log-linear link function, link function based on Box-Cox transformation was studied by Fan et al. [71] under the same setup. Log-linear link function was also used by Xiong [136] and Xiong and Milliken [138] under the assumption of exponentially distributed failure times. However, Watkins [132] argued that it would be better to work with original exponential parameters in case of simple SSLT. Analysis of grouped data under CS-I and PCS-I were addressed by Xiong and Ji [137] and Wu et al. [134]. Miller and Nelson [109], Bai et al. [7], Gouno et al. [75], Han et al. [79], Ebraheim and Al-Masri [66], Balakrishnan and Han [18], and Wu et al. [134] are references on optimality of the SSLT under the assumption of exponential CEM. Step-stress models have been also discussed in Bayesian framework, for example, see Drop et al. [64], Lee and Pan [94]. Optimality of the simple SSLT was considered by Yuan and Liu [141] and Leu and Shen [95] under the Bayesian setup. Xiong and Milliken [138], Xiong et al. [139], Wang and Yu [131], Kateri and Balakrishnan [83], and Kundu and Balakrishnan [89] considered random stress-change time. In these articles it is assumed that the stress levels are changed at time to failure of units. An order restricted inference for exponential step-stress model under CS-I and CS-II can be found in Balakrishnan et al. [13]. Guan and Tang [76] considered multivariate exponential distribution under step-stress model with CS-I. A sequential order statistics approach to SSLTs was considered by Balakrishnan et al. [22]. Balakrishnan [10] provided a nice review of the step-stress model under the assumption of exponential failure data. Log-normally distributed failure time data was considered under step-stress accelerated life tests by Chung and Bai [52], Alhadeed and Yang [3], Balakrishnan et al. [37] and Lin and Chou [96]. Inference for a simple step-stress model with CS-II and Weibull distributed lifetimes can be found in Kateri and Balakrishnan [83]. Properties of Weibull CEM was studied by Komori [86]. Optimal step-stress plan for Weibull distribution and CS-I was considered by Bai and Kim [6]. A new model for Weibull distributed lifetimes was considered by Khamis and Higgins [85]. This model assumes that the power transform of the lifetimes follows a exponential CEM. However, Xu and Tang [140] argued that this model is actually a special case of tampered failure rate model proposed by Bhattacharyya and Soejoeti [42]. Liu [100] considered step-stress model for Weibull distributed lifetimes under Bayesian setup. Estimation of the parameters of generalized exponential distribution was considered by Ismail [81] in presence of PCS-II with random removals. It is worth mentioning here that inference for specific step-stress models has also been discussed in the general framework of accelerated life testing; see, for example Shaked and Singpurwala [127], McNichols and Padgett [106], Lu and Storer [101], Drop and Mazzuchi [63], and Bagdonavicius et al. [5]. Almost all the analyses of the step-stress models have been performed based on the single experiment and a very little effort has been devoted to develop the analysis of multi-sample experiments. Analysis of the step-stress model under the multi-sample experiment can be found in Balakrishnan and Kamps [21] and Kateri et al. [84].

1.6 Organization of the Dissertation

In Chapter 2, we have addressed the problem of estimation of model parameters of two-parameter exponential distribution. The distributions of maximum likelihood estimators of scale and location parameters have been derived. We have found that the distribution of the location parameter is same of that of the lowest order statistic, which has been extensively studied and hence is not addressed further in this dissertation. Based on the assumption of the monotonicity of the cumulative distribution function of the maximum likelihood estimator of the scale parameters, approximate confidence interval of the scale parameter has been derived in this dissertation. Though we could not prove this monotonicity assumption formally, the same has been verified on the basis of extensive numerical simulations. Percentile and bias-corrected Bootstrap confidence intervals have also been considered. Simulations have been done to judge the performance of the different confidence intervals. Analysis of a data set has been performed for illustrative purposes.

In Chapter 3, an attempt has been made to address the Bayesian inference of the unknown parameters of a two-parameter exponential distribution when the data are obtained from different HCSs and PCSs. We notice that the Bayes estimate and credible interval of some parametric function cannot be found in explicit form in general. A simulation based procedure has been proposed to compute Bayes estimate as well as to construct credible intervals. Extensive simulation study has been carried out to understand the effectiveness of the proposed procedure. Analysis of a data set has been performed for illustrative purposes.

We address order restricted Bayesian inference of the unknown parameters of a simple step-stress model under different censoring schemes when the lifetimes of the experimental units are assumed to be exponentially distributed in Chapter 4. We have assumed fairly flexible priors on the unknown parameters. It is observed that in all the cases the Bayes estimates of the unknown parameters cannot be obtained in explicit form. We propose to use the importance sampling technique to compute Bayes estimate and also to construct associated CRI. Extensive Monte Carlo simulations are performed to see the effectiveness of the proposed method in case of CS-I, and the performances are quite satisfactory. The analysis of two data sets have been performed for illustrative purposes.

In Chapter 5, an attempt has been made to address the same issue as in Chapter 4. We analyze a simple step-stress model based on the assumption that the lifetime of the experimental units follows two-parameter exponential distribution. The analysis has been performed based on the assumption that the model satisfies

CEM assumption, and the data are Type-II censored. One of the justifications for incorporating the location parameter is the presence of possible bias in the experimental data due to calibration. It is observed that MLEs of unknown parameters do not always exist. Whenever they exist, they can be obtained in closed form. We obtain the exact conditional distributions of the MLEs of the scale parameters. Fisher information matrix has been used to construct the asymptotic CIs of the unknown scale parameters, assuming the location parameter is known. Parametric bootstrap method has been also used for constructing CI for the scale parameters, and it is very easy to implement in practice. Extensive simulations have been performed to compare the performances of the different methods. One data analysis has been performed for illustrative purposes.

In Chapter 6, we consider a simple step-stress model when the lifetimes are assumed to have two-parameter Weibull distribution. The analysis has been performed under the assumption that the model satisfies KHM assumptions. We have assumed quite flexible priors on the parameters. It has been noticed that the Bayes estimators do not exist in close form in the most cases. Therefore an importance simulation based procedure has been proposed to calculate Bayes estimate and to construct Bayesian credible interval in both the ordered restricted and unrestricted cases. Extensive simulations have been carried out to examine performance of the proposed methods. Analysis of a data set has been also performed for illustrative purpose.

Finally, this dissertation has been concluded in the Chapter 7. We have also pointed out some of the future research directions in the same chapter.

Chapter 2

Exact Inference for the Two-parameter Exponential Distribution under Type-II Hybrid Censoring[1]

2.1 Introduction

Childs et al. [49] have considered the MLEs of the unknown parameters of a two-parameter exponential model, when the data are hybrid Type-I censored (see Section 1.2). The exact distribution of the MLEs of unknown parameters and of the quantiles are provided in this article. They have also discussed different methods for constructing confidence intervals of the unknown parameters and compared them based on extensive simulation study.

A natural extension to Childs et al. [49] is to consider HCS-II (see Section 1.2) under the same setup. The purpose of this chapter is to consider a life test where n items are put on the test under a HCS-II. It is assumed that the lifetime distribution of the experimental units are *i.i.d.* as two-parameter exponential distribution, *i.e.*, PDF of the lifetime of an experimental unit, for $\theta > 0$ and $-\infty < \mu < \infty$, is given

[1] A part of this work has been published in *Journal of Statistical Planning and Inference*, vol. 142, 613–625, 2012.

by

$$f(t;\mu,\theta) = \frac{1}{\theta}\, e^{-\frac{t-\mu}{\theta}} \quad \text{if} \quad t > \mu, \tag{2.1}$$

and 0 otherwise. First we obtain the MLEs of the unknown parameters $\mu \in \mathbb{R}$ and $\theta > 0$, and provide the joint MGF. Based on the joint MGF, we obtain the marginal MGFs, and the marginal distribution functions of the MLEs. From the marginal distribution functions, using the same idea as in Chen and Bhattacharya [47], the confidence interval of θ is obtained by solving two non-linear equations. Since the confidence interval based on the MLEs is quite difficult to implement, we have proposed to use bootstrap confidence intervals, whose implementation are quite straight forward.

Rest of the chapter is organized as follows. In Section 2.2, we present the MLEs of the unknown parameters, and the joint and marginal moment generating functions of the MLEs are derived in Section 2.3. Different confidence intervals are proposed in Section 2.4. In Section 2.5, we consider the approximation of the original PDF of the MLE of θ by gamma PDF using similar method as in Kundu [88]. Simulation results and the analysis of a data set are presented in Section 2.6 and Section 2.7, respectively. Finally we conclude the chapter in Section 2.8. Proofs of all the theorems are provided in the Appendix 2.B.

2.2 Maximum Likelihood Estimators

We have already discussed HCS-II in the Section 1.2.2. Recall that the available data under HCS-II can be one of the from given below.

(a) $t_{1:n} < t_{2:n} < \cdots < t_{r:n}$ if $\tau^* = t_{r:n}$,

(b) $t_{1:n} < t_{2:n} < \cdots < t_{N:n}$ if $\tau^* = \tau$, where N is the number of units failed before the time τ. The likelihood of the observed data is given by

$$L\left(\mu,\,\theta\,|\,\text{Data}\right) = \begin{cases} \frac{n!}{(n-N)!\,\theta^N}\, e^{-\frac{1}{\theta}\sum_{i=1}^{N}(t_{i:n}-\mu)-\frac{1}{\theta}(n-N)(\tau-\mu)} & \text{if } t_{r:n} < \tau \\ \\ \frac{n!}{(n-r)!\,\theta^r}\, e^{-\frac{1}{\theta}\sum_{i=1}^{r}(t_{i:n}-\mu)-\frac{1}{\theta}(n-r)(t_{r:n}-\mu)} & \text{if } t_{r:n} > \tau. \end{cases} \tag{2.2}$$

For $r = 1$ and $N = 0$

$$L(\mu,\,\theta\,|\,\text{Data}) = \frac{n}{\theta}\, e^{-\frac{n}{\theta}(t_{1:n}-\mu)} \quad \text{if } \mu < t_{1:n},\, \theta > 0,$$

and this likelihood function get maximized at $\mu = t_{r:n}$ for fixed θ.

$$L(t_{1:n},\,\theta\,|\,\text{Data}) = \frac{n}{\theta} \quad \text{if } \theta > 0,$$

which increases as θ decreases. Hence, there exists a path along which $L(\mu,\,\theta|\text{Data})$ is unbounded and MLE of $(\mu,\,\theta)$ does not exist for $r = 1$ and $N = 0$. For $r = 1$, the likelihood function in (2.2) possesses its maximum at $(\widehat{\mu},\,\widehat{\theta})$ conditioning on the event $\{N \geq 1\}$, where

$$\widehat{\mu} = t_{1:n} \qquad \text{and} \qquad \widehat{\theta} = \frac{1}{N}\left\{\sum_{i=1}^{N} t_{i:n} - n\,t_{1:n} + (n-N)\,\tau\right\}. \tag{2.3}$$

Hence, $(\widehat{\mu},\,\widehat{\theta})$ is the conditional MLE of $(\mu,\,\theta)$, conditioning on the event $\{N \geq 1\}$ when $r = 1$. For $r \geq 2$, MLE of $(\mu,\,\theta)$ exists for all values of N and is given by $(\widehat{\mu},\,\widehat{\theta})$, where

$$\widehat{\mu} = t_{1:n} \qquad \text{and} \qquad \widehat{\theta} = \frac{1}{N^*}\left\{\sum_{i=1}^{N^*} t_{i:n} - n\,t_{1:n} + (n-N^*)\,\tau^*\right\} \tag{2.4}$$

with $\tau^* = \max\{t_{r:n},\,\tau\}$ and $N^* = \max\{r,\,N\}$.

2.3 Joint and Marginal MGF

We have already seen that MLE of $(\mu,\,\theta)$ does not exist if $r = 1$ and $N = 0$. Hence, for $r = 1$, we consider the conditional distribution of MLEs of the unknown parameters conditioning on the event $\{N \geq 1\}$. For $r \geq 2$, MLE of $(\mu,\,\theta)$ exists for all values of N. In order to find the distribution of $\widehat{\mu}$ and $\widehat{\theta}$, we first find the joint

MGF of $(\widehat{\mu}, \widehat{\theta})$ and then invert it to get the distribution of the MLEs as it was first suggested by Bartholmew [39].

Theorem 2.3.1. For $r = 1$, conditional joint MGF of $(\widehat{\mu}, \widehat{\theta})$ conditioning on the event $\{N \geq 1\}$ exists for all $-\infty < \omega_1 < \infty$ and $-\infty < \omega_2 < \infty$ and is given by

$$
\begin{aligned}
&E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}} \mid N \geq 1] \\
&= (1-q^n)^{-1} \left[c_{10} \frac{e^{\mu_{10}\omega_1 + \mu\,\omega_2}}{\left(1 + \frac{\omega_1}{\lambda_{10}} - \frac{\omega_2}{\nu_0}\right)} - d_{10} \frac{e^{\tau\omega_2}}{\left(1 + \frac{\omega_1}{\lambda_{10}} - \frac{\omega_2}{\nu_0}\right)} + \frac{e^{\mu\,\omega_2} - q^n e^{\tau\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_n}\right)^{\alpha_n} \left(1 - \frac{\omega_2}{\nu_{n-1}}\right)} \right. \\
&\quad + \sum_{i=2}^{n-1} \sum_{j=0}^{i-1} \left\{ c_{ij} \frac{e^{\mu_{ij}\omega_1 + \mu\,\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_i}\right)^{\alpha_i} \left(1 + \frac{\omega_1}{\lambda_{ij}} - \frac{\omega_2}{\nu_j}\right)} - d_{ij} \frac{e^{\tau\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_i}\right)^{\alpha_i} \left(1 + \frac{\omega_1}{\lambda_{ij}} - \frac{\omega_2}{\nu_j}\right)} \right\} \\
&\quad \left. + \sum_{j=0}^{n-2} \left\{ c_{nj} \frac{e^{\mu_{nj}\omega_1 + \mu\,\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_n}\right)^{\alpha_n} \left(1 + \frac{\omega_1}{\lambda_{nj}} - \frac{\omega_2}{\nu_j}\right)} - d_{nj} \frac{e^{\tau\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_n}\right)^{\alpha_n} \left(1 + \frac{\omega_1}{\lambda_{nj}} - \frac{\omega_2}{\nu_j}\right)} \right\} \right].
\end{aligned}
\tag{2.5}
$$

For $r \geq 2$ the joint MGF of $(\widehat{\mu}, \widehat{\theta})$ exists for $\omega_1 < \dfrac{r}{\theta}$ and $\omega_2 < \dfrac{1}{\theta} + \dfrac{n-1}{r}\,\omega_1$ and is given by

$$
\begin{aligned}
&E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}}] \\
&= \frac{q^n\, e^{\tau\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_1}\right)^{\alpha_1} \left(1 - \frac{\omega_2}{\nu_{n-1}}\right)} + \frac{e^{\mu\,\omega_2} - q^n e^{\tau\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_n}\right)^{\alpha_n} \left(1 - \frac{\omega_2}{\nu_{n-1}}\right)} \\
&\quad + \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} c_{ij} \frac{e^{\mu_{ij}\,\omega_1 + \mu\,\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_i}\right)^{\alpha_i} \left(1 + \frac{\omega_1}{\lambda_{ij}} - \frac{\omega_2}{\nu_j}\right)} - \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} d_{ij} \frac{e^{\tau\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_i}\right)^{\alpha_i} \left(1 + \frac{\omega_1}{\lambda_{ij}} - \frac{\omega_2}{\nu_j}\right)} \\
&\quad + \sum_{j=0}^{n-2} c_{nj} \frac{e^{\mu_{nj}\,\omega_1 + \mu\,\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_n}\right)^{\alpha_n} \left(1 + \frac{\omega_1}{\lambda_{nj}} - \frac{\omega_2}{\nu_j}\right)} - \sum_{j=0}^{n-2} d_{nj} \frac{e^{\tau\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_n}\right)^{\alpha_n} \left(1 + \frac{\omega_1}{\lambda_{nj}} - \frac{\omega_2}{\nu_j}\right)}
\end{aligned}
\tag{2.6}
$$

when $\mu < \tau$ and for $\mu \geq \tau$

$$
E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}}] = \frac{e^{\mu\,\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_1}\right)^{\alpha_1} \left(1 - \frac{\omega_2}{\nu_{n-1}}\right)}, \tag{2.7}
$$

where

$$
\begin{aligned}
q &= e^{-\frac{\tau-\mu}{\theta}},\\
\nu_i &= \frac{i+1}{\theta} I_{A_{(0,n-1)}}(i),\\
\lambda_i &= \frac{r}{\theta} I_{A_{(1,r-1)}}(i) + \frac{i}{\theta} I_{A_{(r,n)}}(i),\\
\alpha_i &= (r-1) I_{A_{(1,r-1)}}(i) + (i-1) I_{A_{(r,n)}}(i),\\
\mu_{ij} &= \frac{1}{r}\left(n-j-1\right)\left(\tau-\mu\right) I_{A_{(0,i-1)}}(j) I_{A_{(1,r-1)}}(i)\\
&\quad + \frac{1}{i}\left(n-j-1\right)\left(\tau-\mu\right) I_{A_{(0,i-1)}}(j) I_{A_{(r,n)}}(i),\\
\lambda_{ij} &= \frac{r\left(j+1\right)}{\left(n-j-1\right)\theta} I_{A_{(0,i-1)}}(j) I_{A_{(1,r-1)}}(i) + \frac{i\left(j+1\right)}{\left(n-j-1\right)\theta} I_{A_{(0,i-1)}}(j) I_{A_{(r,n)}}(i),\\
c_{ij} &= (-1)^{i-j-1}\binom{n}{i}\binom{i}{j+1} q^{n-j-1} I_{A_{(0,i-1)}}(j) I_{A_{(1,n)}}(i),\\
d_{ij} &= (-1)^{i-j-1}\binom{n}{i}\binom{i}{j+1} q^{n} I_{A_{(0,i-1)}}(j) I_{A_{(1,n)}}(i),\\
A_{(p,q)} &= \{p,\, p+1,\, \cdots,\, q\} \text{ for } p<q,\\
I_A(x) &= \begin{cases} 1 & \text{if } x \in A\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

∎

Remark 2.3.1. Note that as $\tau \to \infty$, the joint MGF of $\widehat{\theta}$ and $\widehat{\mu}$ at (ω_1, ω_2) reduces to

$$
\frac{e^{\mu\,\omega_2}}{\left(1-\frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1-\frac{\omega_2}{\nu_{n-1}}\right)},
$$

which is the joint MGF of $\widehat{\theta}$ and $\widehat{\mu}$ in case of complete sample. That means as $\tau \to \infty$, $2n\widehat{\theta}/\theta$ is distributed as χ^2 random variable with $2n-2$ degrees of freedom, $n(\widehat{\mu}-\mu)/\theta$ is a standard exponential random variable and they are independently distributed. ∎

Remark 2.3.2. For $r \geq 2$, if $\tau \leq \mu$ the joint MGF of $\widehat{\theta}$ and $\widehat{\mu}$ at (ω_1, ω_2) is given in (2.7), which is the joint MGF of $\widehat{\theta}$ and $\widehat{\mu}$ in case of ordinary Type-II censoring scheme. In this case $2r\widehat{\theta}/\theta$ is distributed as χ^2 random variable with $2r-2$ degrees of freedom, $n(\widehat{\mu}-\mu)/\theta$ is a standard exponential random variable and they are independently distributed. ∎

Corollary 2.3.1. The marginal MGF of $\widehat{\theta}$ for $r=1$ exists for all $-\infty < \omega_1 < \infty$ and is given by

$$
\begin{aligned}
&E[e^{\omega_1 \widehat{\theta}} \,|\, N \geq 1] \\
&= \frac{1}{\left(1-\frac{\omega_1}{\lambda_n}\right)^{\alpha_n}} + (1-q^n)^{-1}\left[c_{10}\,\frac{e^{\mu_{10}\omega_1}}{\left(1+\frac{\omega_1}{\lambda_{10}}\right)} - d_{10}\,\frac{1}{\left(1+\frac{\omega_1}{\lambda_{10}}\right)}\right. \\
&\quad + \sum_{i=2}^{n-1}\sum_{j=0}^{i-1}\left\{c_{ij}\,\frac{e^{\mu_{ij}\omega_1}}{\left(1-\frac{\omega_1}{\lambda_i}\right)^{\alpha_i}\left(1+\frac{\omega_1}{\lambda_{ij}}\right)} - d_{ij}\,\frac{1}{\left(1-\frac{\omega_1}{\lambda_i}\right)^{\alpha_i}\left(1+\frac{\omega_1}{\lambda_{ij}}\right)}\right\} \\
&\quad \left. + \sum_{j=0}^{n-2}\left\{c_{nj}\,\frac{e^{\mu_{nj}\omega_1}}{\left(1-\frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1+\frac{\omega_1}{\lambda_{nj}}\right)} - d_{nj}\,\frac{1}{\left(1-\frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1+\frac{\omega_1}{\lambda_{nj}}\right)}\right\}\right].
\end{aligned}
$$

For $r \geq 2$, MGF of $\widehat{\theta}$ exists when $-\frac{r}{(n-1)\theta} < \omega_1 < \frac{r}{\theta}$ and is

$$
\begin{aligned}
E[e^{\omega_1 \widehat{\theta}}] &= \frac{q^n}{\left(1-\frac{\omega_1}{\lambda_1}\right)^{\alpha_1}} + \frac{1-q^n}{\left(1-\frac{\omega_1}{\lambda_n}\right)^{\alpha_n}} \\
&\quad + \sum_{i=1}^{n-1}\sum_{j=0}^{i-1} c_{ij}\,\frac{e^{\mu_{ij}\,\omega_1}}{\left(1-\frac{\omega_1}{\lambda_i}\right)^{\alpha_i}\left(1+\frac{\omega_1}{\lambda_{ij}}\right)} - \sum_{i=1}^{n-1}\sum_{j=0}^{i-1} d_{ij}\,\frac{1}{\left(1-\frac{\omega_1}{\lambda_i}\right)^{\alpha_i}\left(1+\frac{\omega_1}{\lambda_{ij}}\right)} \\
&\quad + \sum_{j=0}^{n-2} c_{nj}\,\frac{e^{\mu_{nj}\,\omega_1}}{\left(1-\frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1+\frac{\omega_1}{\lambda_{nj}}\right)} - \sum_{j=0}^{n-2} d_{nj}\,\frac{1}{\left(1-\frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1+\frac{\omega_1}{\lambda_{nj}}\right)},
\end{aligned}
$$

when $\mu < \tau$ and for $\mu \geq \tau$

$$
E[e^{\omega_1 \widehat{\theta}}] = \frac{1}{\left(1-\frac{\omega_1}{\lambda_1}\right)^{\alpha_1}}.
$$
∎

Corollary 2.3.2. The marginal MGF of $\widehat{\mu}$ for $r=1$ exists for all $-\infty < \omega_2 < \infty$ and it is given by

$$
\begin{aligned}
&E[e^{\omega_2 \widehat{\mu}} \mid N \geq 1] \\
&= (1-q^n)^{-1} \left[c_{10} \frac{e^{\mu \omega_2}}{\left(1-\frac{\omega_2}{\nu_0}\right)} - d_{10} \frac{e^{\tau\omega_2}}{\left(1-\frac{\omega_2}{\nu_0}\right)} + \frac{e^{\mu \omega_2}}{\left(1-\frac{\omega_2}{\nu_{n-1}}\right)} - q^n \frac{e^{\tau\omega_2}}{\left(1-\frac{\omega_2}{\nu_{n-1}}\right)} \right. \\
&\left. + \sum_{i=2}^{n-1}\sum_{j=0}^{i-1} \left\{ c_{ij} \frac{e^{\mu \omega_2}}{\left(1-\frac{\omega_2}{\nu_j}\right)} - d_{ij} \frac{e^{\tau\omega_2}}{\left(1-\frac{\omega_2}{\nu_j}\right)} \right\} + \sum_{j=0}^{n-2} \left\{ c_{nj} \frac{e^{\mu \omega_2}}{\left(1-\frac{\omega_2}{\nu_j}\right)} - d_{nj} \frac{e^{\tau\omega_2}}{\left(1-\frac{\omega_2}{\nu_j}\right)} \right\} \right].
\end{aligned}
$$

For $r \geq 2$, the MGF of $\widehat{\mu}$ for $\omega_2 < \frac{1}{\theta}$ is given by

$$
\begin{aligned}
E[e^{\omega_2 \widehat{\mu}}] = \frac{e^{\mu \omega_2}}{\left(1-\frac{\omega_2}{\nu_{n-1}}\right)} &+ \sum_{i=1}^{n-1}\sum_{j=0}^{i-1} \left\{ c_{ij} \frac{e^{\mu \omega_2}}{\left(1-\frac{\omega_2}{\nu_j}\right)} - d_{ij} \frac{e^{\tau\omega_2}}{\left(1-\frac{\omega_2}{\nu_j}\right)} \right\} \\
&+ \sum_{j=0}^{n-2} \left\{ c_{nj} \frac{e^{\mu \omega_2}}{\left(1-\frac{\omega_2}{\nu_j}\right)} - d_{nj} \frac{e^{\tau\omega_2}}{\left(1-\frac{\omega_2}{\nu_j}\right)} \right\}
\end{aligned}
$$

when $\mu < \tau$ and for $\mu \geq \tau$

$$
E[e^{\omega_2 \widehat{\mu}}] = \frac{e^{\mu \omega_2}}{\left(1-\frac{\omega_2}{\nu_{n-1}}\right)}. \qquad \blacksquare
$$

From the MGF of $\widehat{\theta}$, the PDF of $\widehat{\theta}$ can be obtained by using the inversion technique as suggested by Chen and Bhattacharya [47]. The details are available in Appendices 2.A and 2.B.

Theorem 2.3.2. For $r = 1$, conditional PDF of $\widehat{\theta}$, conditioning on the event $\{N \geq 1\}$, for $-\infty < t < \infty$ is

$$
\begin{aligned}
f_{\widehat{\theta}}(t) = g_1(t; \alpha_n, \lambda_n) + (1-q^n)^{-1} &\Big[c_{10}\, g_1(-t+\mu_{10}; 1, \lambda_{10}) - d_{10}\, g_1(-t; 1, \lambda_{10}) \\
&+ \sum_{i=2}^{n-1}\sum_{j=0}^{i-1} \{ c_{ij}\, g_2(t-\mu_{ij}; \alpha_i, \lambda_i, \lambda_{ij}) - d_{ij}\, g_2(t; \alpha_i, \lambda_i, \lambda_{ij}) \} \\
&+ \sum_{j=0}^{n-2} \{ c_{nj}\, g_2(t-\mu_{nj}; \alpha_n, \lambda_n, \lambda_{nj}) - d_{nj}\, g_2(t; \alpha_n, \lambda_n, \lambda_{nj}) \} \Big]. \qquad (2.8)
\end{aligned}
$$

For $r \geq 2$, the PDF of $\widehat{\theta}$ for $-\infty < t < \infty$ is

$$f_{\widehat{\theta}}(t) = \begin{cases} g_3(t) & \text{if } \mu < \tau \\ \\ g_1\left(t; \alpha_1, \lambda_1\right) & \text{if } \mu \geq \tau, \end{cases} \tag{2.9}$$

where

$$\begin{aligned}
g_1\left(t; \alpha, \lambda\right) &= \frac{\lambda^{\alpha}}{\Gamma\left(\alpha\right)}\, e^{-\lambda t}\, t^{\alpha-1}\, I_{(0,\infty)}(t), \quad t \in \mathbb{R}, \\
g_2\left(t; \alpha, \lambda_1, \lambda_2\right) &= \sum_{k=0}^{\alpha-1} p_k\, g_1\left(t; \alpha-k, \lambda_1\right) + \left(1 - \sum_{k=0}^{\alpha-1} p_k\right) g_1\left(-t, 1, \lambda_2\right), \quad t \in \mathbb{R}, \\
p_k &= \frac{\lambda_2}{\lambda_1+\lambda_2}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k, \quad k = 0,\, 1,\, \cdots,\, \alpha-1, \\
g_3(t) &= q^n\, g_1\left(t; \alpha_1, \lambda_1\right) + \left(1-q^n\right) g_1\left(t; \alpha_n, \lambda_n\right) \\
&\quad + \sum_{i=1}^{n-1}\sum_{j=0}^{i-1}\left\{c_{ij}\, g_2\left(t-\mu_{ij}; \alpha_i, \lambda_i, \lambda_{ij}\right) - d_{ij}\, g_2\left(t; \alpha_i, \lambda_i, \lambda_{ij}\right)\right\} \\
&\quad + \sum_{j=0}^{n-2}\left\{c_{nj}\, g_2\left(t-\mu_{nj}; \alpha_n, \lambda_n, \lambda_{nj}\right) - d_{nj}\, g_2\left(t; \alpha_n, \lambda_n, \lambda_{nj}\right)\right\}, \quad t \in \mathbb{R}. \blacksquare
\end{aligned}$$

Note that $g_3(t)$ depends on μ_{ij}, α_i, λ_i and λ_{ij}; $i, j = 1, \ldots, n$, for brevity we do not write it explicitly. From the Theorem 2.3.2, since the integration of density function over the whole range is one, we have the following identity

$$\sum_{i=1}^{n-1}\sum_{j=0}^{i-1}\left(c_{ij} - d_{ij}\right) + \sum_{j=0}^{n-2}\left(c_{nj} - d_{nj}\right) = 0.$$

Theorem 2.3.3. When $-\infty < t < \infty$, for $r = 1$, the conditional PDF of $\widehat{\mu}$ conditioning on $\{N \geq 1\}$ is given by

$$\begin{aligned}
f_{\widehat{\mu}}\left(t\right) = \left(1-q^n\right)^{-1}\Bigg[& c_{10}\, g_1\left(t-\mu; 1, \nu_0\right) - d_{10} g_1\left(t-\tau; 1, \nu_0\right) + g_1\left(t-\mu; 1, \nu_{n-1}\right) \\
& - q^n g_1\left(t-\tau; 1, \nu_{n-1}\right) + \sum_{i=2}^{n-1}\sum_{j=0}^{i-1}\left\{c_{ij}\, g_1\left(t-\mu; 1, \nu_j\right) - d_{ij}\, g_1\left(t-\tau; 1, \nu_j\right)\right\} \\
& + \sum_{j=0}^{n-2}\left\{c_{nj}\, g_1\left(t-\mu; 1, \nu_j\right) - d_{nj}\, g_1\left(t-\tau; 1, \nu_j\right)\right\}\Bigg],
\end{aligned}$$

and for $r \geq 2$ the PDF of $\widehat{\mu}$ is

$$f_{\widehat{\mu}}(t) = g_4(t-\mu)\, I_{(-\infty,\tau)}(\mu) + g_1(t-\mu; 1, \nu_{n-1})\; I_{(\tau,\infty)}(\mu),$$

where $g_1(\cdot)$ is as previous Theorem and

$$g_4(t) = g_1(t; 1, \nu_{n-1}) + \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} \{c_{ij}\, g_1(t; 1, \nu_j) - d_{ij}\, g_1(t+\mu-\tau; 1, \nu_j)\}$$
$$+ \sum_{j=0}^{n-2} \{c_{nj}\, g_1(t; 1, \nu_j) - d_{nj}\, g_1(t+\mu-\tau; 1, \nu_j)\}. \qquad \blacksquare$$

From the PDF of $\widehat{\theta}$, the corresponding moments can be easily obtained. The first two moments of $\widehat{\theta}$ are as follows:

For $r = 1$,

$$E[\widehat{\theta}] = \theta + \theta A_1(\mu,\theta) + (1-q^n)^{-1} B_1(\mu,\theta),$$
$$E[\widehat{\theta}^2] = \theta^2 + \theta^2 C_1(\mu,\theta) + \theta D_1(\mu,\theta) + (1-q^n)^{-1} E_1(\mu,\theta).$$

For $r \geq 2$,

$$E[\widehat{\theta}] = \begin{cases} \theta + \theta A_2(\mu,\theta) + B_1(\mu,\theta) & \text{if } \mu < \tau \\ \left(1 - \dfrac{1}{r}\right)\theta & \text{if } \mu \geq \tau, \end{cases}$$

$$E[\widehat{\theta}^2] = \begin{cases} \theta^2 + \theta^2 C_2(\mu,\theta) + \theta D_2(\mu,\theta) + E_1(\mu,\theta) & \text{if } \mu < \tau \\ \left(1 - \dfrac{1}{r}\right)\theta^2 & \text{if } \mu \geq \tau, \end{cases}$$

where

$$A_1(\mu,\theta) = \Bigg[\sum_{i=2}^{n} \sum_{\substack{j=0 \\ j\neq n-1}}^{i-1} (c_{ij} - d_{ij}) \left\{\sum_{k=0}^{i-2} p_{jk} \frac{i-k-1}{i} - \left(1 - \sum_{k=0}^{i-2} p_{jk}\right) \frac{n-j-1}{i(j+1)}\right\}$$
$$- \frac{1}{r}(c_{10} - d_{10})(n-1) - \frac{1-q^n}{n}\Bigg](1-q^n)^{-1},$$

$$B_1(\mu,\theta) = \sum_{i=1}^{n} \sum_{j=0}^{i-1} c_{ij}\mu_{ij},$$

$$
\begin{aligned}
C_1(\mu,\theta) =& \Bigg[\sum_{i=1}^{n} \sum_{\substack{j=0 \\ j\neq n-1}}^{i-1} (c_{ij}-d_{ij}) \left\{ \sum_{k=0}^{i-2} p_{jk} \frac{i-k-1}{(i-k)^{-1} i^2} + 2\left(1-\sum_{k=0}^{i-2} p_{jk}\right) \frac{n-j-1}{i^2(j+1)^2} \right\} \\
&+ 2(n-1)(c_{10}-d_{10}) - \frac{1-q^n}{n} \Bigg] (1-q^n)^{-1},
\end{aligned}
$$

$$
\begin{aligned}
D_1(\mu,\theta) =& \frac{2}{1-q^n} \Bigg[\sum_{i=1}^{n} \sum_{\substack{j=0 \\ j\neq n-1}}^{i-1} c_{ij}\mu_{ij} \left\{ \sum_{k=0}^{i-2} p_{jk} \frac{i-k-1}{i} \right. \\
&\left. - \left(1-\sum_{k=0}^{i-2} p_{jk}\right) \frac{n-j-1}{i(j+1)} \right\} + (n-1)c_{10}\mu_{10} \Bigg],
\end{aligned}
$$

$$
E_1(\mu,\theta) = \sum_{i=1}^{n} \sum_{j=0}^{i-1} c_{ij}\mu_{ij}^2,
$$

$$
\begin{aligned}
A_2(\mu,\theta) =& q^n\left(\frac{1}{n}-\frac{1}{r}\right) - \frac{1}{n} \\
&+ \sum_{i=1}^{r-1}\sum_{j=0}^{i-1} (c_{ij}-d_{ij}) \left\{ \sum_{k=0}^{r-2} p_{jk} \frac{r-k-1}{r} - \left(1-\sum_{k=0}^{r-2} p_{jk}\right) \frac{n-j-1}{r(j+1)} \right\} \\
&+ \sum_{i=r}^{n} \sum_{\substack{j=0 \\ j\neq n-1}}^{i-1} (c_{ij}-d_{ij}) \left\{ \sum_{k=0}^{i-2} p_{jk} \frac{i-k-1}{i} - \left(1-\sum_{k=0}^{i-2} p_{jk}\right) \frac{n-j-1}{i(j+1)} \right\},
\end{aligned}
$$

$$
\begin{aligned}
C_2(\mu,\theta) =& \sum_{i=1}^{r-1}\sum_{j=0}^{i-1} (c_{ij}-d_{ij}) \left\{ \sum_{k=0}^{r-2} p_{jk} \frac{r-k-1}{r^2(r-k)^{-1}} + 2\left(1-\sum_{k=0}^{r-2} p_{jk}\right) \frac{(n-j-1)^2}{r^2(j+1)^2} \right\} \\
&+ \sum_{i=r}^{n} \sum_{\substack{j=0 \\ j\neq n-1}}^{i-1} (c_{ij}-d_{ij}) \left\{ \sum_{k=0}^{i-2} p_{jk} \frac{(i-k-1)}{i^2(i-k)^{-1}} \right. \\
&\left. +2\left(1-\sum_{k=0}^{i-2} p_{jk}\right) \frac{(n-j-1)^2}{i^2(j+1)^2} \right\} - \frac{q^n}{r} - \frac{1-q^n}{n},
\end{aligned}
$$

$$
\begin{aligned}
D_2(\mu,\theta) =& 2\sum_{i=1}^{r-1}\sum_{j=0}^{i-1} c_{ij}\mu_{ij} \left\{ \sum_{k=0}^{r-2} p_{jk} \frac{r-k-1}{r} - \left(1-\sum_{k=0}^{r-2} p_{jk}\right) \frac{n-j-1}{r(j+1)} \right\} \\
&+ 2\sum_{i=r}^{n} \sum_{\substack{j=0 \\ j\neq n-1}}^{i-1} c_{ij}\mu_{ij} \left\{ \sum_{k=0}^{i-2} p_{jk} \frac{i-k-1}{i} - \left(1-\sum_{k=0}^{i-2} p_{jk}\right) \frac{n-j-1}{i(j+1)} \right\}.
\end{aligned}
$$

The CDFs of $\widehat{\theta}$ and $\widehat{\mu}$ can be easily obtained from their respective PDFs. Let us denote $\max\{0, x\}$ by $\langle x\rangle$. For $t \in \mathbb{R}$, the CDF of $\widehat{\theta}$ for $r = 1$ conditioning on the event $N \geq 1$ is

$$\begin{aligned}
F_{\widehat{\theta}}(t) \;=\; & G_1(\lambda_n\, t, \alpha_n) + (1-q^n)^{-1}\Bigg[c_{10}\, e^{-\lambda_{10}\,\langle \mu_{10}-t\rangle} - d_{10}\, e^{-\lambda_{10}\,\langle -t\rangle} \\
& + \sum_{i=2}^{n-1}\sum_{j=0}^{i-1}\{c_{ij}\, G_2(t;\mu_{ij},\alpha_i,\lambda_i,\lambda_{ij}) - d_{ij}\, G_2\,(t;0,\alpha_i,\lambda_i,\lambda_{ij})\} \\
& + \sum_{j=0}^{n-2}\{c_{nj}\, G_2(t;\mu_{nj},\alpha_n,\lambda_n.\lambda_{nj}) - d_{nj}\, G_2\,(t;0,\alpha_n,\lambda_n,\lambda_{nj})\}\Bigg],
\end{aligned}$$

and for $r \geq 2$, the CDF is

$$F_{\widehat{\theta}}(t) = \begin{cases} q^n\, G_1(\lambda_1\, t,\alpha_1) + (1-q^n)\; G_1(\lambda_n\, t,\alpha_n) & \\ \quad + \sum_{i=2}^{n-1}\sum_{j=0}^{i-1}\{c_{ij}\, G_2(t;\mu_{ij},\alpha_i,\lambda_i,\lambda_{ij}) - d_{ij}\, G_2\,(t;0,\alpha_i,\lambda_i,\lambda_{ij})\} & \\ \quad + \sum_{j=0}^{n-2}\{c_{nj}\, G_2(t;\mu_{nj},\alpha_n,\lambda_n.\lambda_{nj}) - d_{nj}\, G_2\,(t;0,\alpha_n,\lambda_n,\lambda_{nj})\} & \text{if } \mu < \tau \\ G_1(\lambda_1\, t,\alpha_1) & \text{if } \mu \geq \tau, \end{cases}$$

where

$$G_1(t,\alpha) = \begin{cases} 0 & \text{if } t < 0 \\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t w^{\alpha-1}\, e^{-w} dw & \text{if } t \geq 0, \end{cases}$$

and

$$G_2(t;\, \mu,\, \alpha,\, \lambda_1\, \lambda_2) = \begin{cases} \left(1 - \displaystyle\sum_{k=0}^{\alpha-1} p_k\right) e^{\lambda_2 t} & \text{if } t < 0 \\ \displaystyle\sum_{k=0}^{\alpha-1} p_k\, G_1(\lambda_1 t,\, \alpha - k) & \text{if } t \geq 0. \end{cases}$$

The CDF of $\widehat{\mu}$ for $r = 1$ conditioning on the event $N \geq 1$ is

$$F_{\widehat{\mu}}(t) = \begin{cases} 0 & \text{if } t < \mu \\ (1-q^n)^{-1}\left[1 - e^{-\frac{n}{\theta}(t-\mu)}\right] & \text{if } \mu \leq t < \tau \\ 1 & t \geq \tau, \end{cases}$$

and for $r \geq 2$, the same is

$$F_{\widehat{\mu}}(t) = \begin{cases} 0 & \text{if } t < \mu \\ 1 - e^{-\frac{n}{\theta}(t-\mu)} & \text{if } t \geq \mu. \end{cases}$$

Clearly the distribution of $\widehat{\mu}$ is same of that of the first order statistics from an exponential distribution for $r \geq 2$. For $r = 1$, the distribution of $\widehat{\mu}$ is same as the conditional distribution of the first order statistics from an exponential distribution conditioning on the event that it lies in the interval (μ, τ). This statistic has been quite extensively studied in literature, and therefore we do not pursue it in this dissertation.

Note that the p^{th} quantile of the exponential distribution given in (2.1) is $\eta_p = \mu + a_p\,\theta$, where $a_p = -ln(1-p)$. Therefore the MLE of η_p can be found by replacing μ and θ by their respective MLEs. The moment generating function of the MLE of η_p, say $\widehat{\eta}_p$, can be obtained easily from the joint MGF of $\widehat{\mu}$ and $\widehat{\theta}$ in Theorem 2.3.1. The MGF of $\widehat{\eta}_p$ for $r = 1$ exists for all $-\infty < \omega < \infty$ and is

$$\begin{aligned} &E\left(e^{\omega\,\widehat{\eta}_p} | N \geq 1\right) \\ &\quad = (1-q^n)^{-1}\left[\frac{c_{10}\,e^{(\mu+\mu_{10}\,a_p)\,\omega}}{1-\beta_{10}\,\omega} - \frac{d_{10}\,e^{\tau\,\omega}}{1-\beta_{10}\,\omega} + \frac{e^{\mu\,\omega} - q^n\,e^{\tau\,\omega}}{\left(1-\frac{a_p}{\lambda_n}\omega\right)^{\alpha_n}\left(1-\frac{\omega}{\nu_{n-1}}\right)}\right. \\ &\qquad + \sum_{i=2}^{n-1}\sum_{j=0}^{i-1}\left\{\frac{c_{ij}\,e^{(\mu+a_p\,\mu_{ij})\,\omega}}{\left(1-\frac{a_p}{\lambda_i}\,\omega\right)^{\alpha_i}(1-\beta_{ij}\,\omega)} - \frac{d_{ij}\,e^{\tau\,\omega}}{\left(1-\frac{a_p}{\lambda_i}\,\omega\right)^{\alpha_i}(1-\beta_{ij}\,\omega)}\right\} \\ &\qquad \left. + \sum_{j=0}^{n-2}\left\{\frac{c_{nj}\,e^{(\mu+a_p\,\mu_{nj})\,\omega}}{\left(1-\frac{a_p}{\lambda_n}\,\omega\right)^{\alpha_n}(1-\beta_{nj}\,\omega)} - \frac{d_{nj}\,e^{\tau\,\omega}}{\left(1-\frac{a_p}{\lambda_n}\,\omega\right)^{\alpha_n}(1-\beta_{nj}\,\omega)}\right\}\right]. \end{aligned}$$

For $r \geq 2$ the MGF of $\widehat{\eta}_p$ exists at least for ω in some neighborhood of zero and is given by

$$\begin{aligned} E[e^{\omega\,\widehat{\eta}_p}] =& \frac{q^n\,e^{\tau\,\omega}}{\left(1-\frac{a_p}{\lambda_1}\,\omega\right)^{\alpha_1}\left(1-\frac{1}{\nu_{n-1}}\,\omega\right)} + \frac{e^{\mu\,\omega} - q^n\,e^{\tau\,\omega}}{\left(1-\frac{a_p}{\lambda_n}\,\omega\right)^{\alpha_n}\left(1-\frac{1}{\nu_{n-1}}\,\omega\right)} \\ &+ \sum_{i=1}^{n-1}\sum_{j=0}^{i-1}\left\{c_{ij}\,\frac{e^{(\mu+a_p\,\mu_{ij})\,\omega}}{\left(1-\frac{a_p}{\lambda_i}\,\omega\right)^{\alpha_i}(1-\beta_{ij}\,\omega)} - d_{ij}\,\frac{e^{\tau\,\omega}}{\left(1-\frac{a_p}{\lambda_i}\,\omega\right)^{\alpha_i}(1-\beta_{ij}\,\omega)}\right\} \end{aligned}$$

$$+\sum_{j=0}^{n-1}\left\{c_{nj}\,\frac{e^{(\mu+a_p\,\mu_{nj})\,\omega}}{\left(1-\frac{a_p}{\lambda_n}\,\omega\right)^{\alpha_n}(1-\beta_{nj}\,\omega)}-d_{nj}\,\frac{e^{\tau\,\omega}}{\left(1-\frac{a_p}{\lambda_n}\,\omega\right)^{\alpha_n}(1-\beta_{nj}\,\omega)}\right\},$$

when $\mu < \tau$ and when $\mu \geq \tau$

$$E[e^{\omega\,\widehat{\eta}_p}] = \frac{e^{\mu\,\omega}}{\left(1-\frac{a_p}{\lambda_1}\,\omega\right)^{\alpha_1}\left(1-\frac{1}{\nu_{n-1}}\,\omega\right)},$$

where $\beta_{ij} = \dfrac{1}{\nu_j} - \dfrac{a_p}{\lambda_{ij}}$ and q, μ_{ij}, α_i, ν_i, λ_i, λ_{ij}, c_{ij} and d_{ij}; $i, j = 1, \ldots, n$ are given in Theorem 2.3.1.

From these MGFs, one can find the PDF of $\widehat{\eta}_p$ using the Lemmas 2.A.3, 2.A.4 and 2.A.5. For $r = 1$, conditional PDF of $\widehat{\eta}_p$, conditioning on the event $\{N \geq 1\}$, at $t \in \mathbb{R}$ is as follows:

$$\begin{aligned}
f_{\widehat{\eta}_p}(t) = \ & (1-q^n)^{-1}\Big[c_{10}\,h_1\,(t-\mu-\mu_{10};\beta_{10}) - d_{10}\,h_1\,(t-\tau;\beta_{10}) \\
& + h_2(t-\mu;\alpha_n,\frac{\lambda_n}{a_p},\frac{1}{\nu_{n-1}}) - q^n\,h_2\left(t-\tau;\alpha_n,\frac{\lambda_n}{a_p},\frac{1}{\nu_{n-1}}\right) \\
& + \sum_{i=2}^{n-1}\sum_{j=0}^{i-1}\left\{c_{ij}\,h_2\left(t-\mu-\mu_{ij}\,a_p;\alpha_i,\frac{\lambda_i}{a_p},\beta_{ij}\right) - d_{ij}\,h_2\left(t;\alpha_i,\frac{\lambda_i}{a_p},\beta_{ij}\right)\right\} \\
& + \sum_{j=0}^{n-2}\left\{c_{nj}\,h_2\left(t-\mu-\mu_{nj}\,a_p;\alpha_n,\frac{\lambda_n}{a_p},\beta_{nj}\right) - d_{nj}\,h_2\left(t;\alpha_n,\frac{\lambda_n}{a_p},\beta_{nj}\right)\right\}\Big].
\end{aligned}$$

For $r \geq 2$, PDF of $\widehat{\eta}_p$ at the point $t \in \mathbb{R}$ when $\mu < \tau$ is

$$\begin{aligned}
f_{\widehat{\eta}_p}(t) = \ & q^n\,h_2\left(t-\tau;\alpha_1,\frac{\lambda_1}{a_p},\frac{1}{\nu_{n-1}}\right) + h_2\left(t-\mu;\alpha_n,\frac{\lambda_n}{a_p},\frac{1}{\nu_{n-1}}\right) \\
& - q^n\,h_2\left(t-\tau;\alpha_n,\frac{\lambda_n}{a_p},\frac{1}{\nu_{n-1}}\right) \\
& + \sum_{i=1}^{n-1}\sum_{j=0}^{i-1}\left\{c_{ij}\,h_2\left(t-\mu-\mu_{ij}\,a_p;\alpha_i,\frac{\lambda_i}{a_p},\beta_{ij}\right) - d_{ij}\,h_2\left(t;\alpha_i,\frac{\lambda_i}{a_p},\beta_{ij}\right)\right\} \\
& + \sum_{j=0}^{n-2}\left\{c_{nj}\,h_2\left(t-\mu-\mu_{nj}\,a_p;\alpha_n,\frac{\lambda_n}{a_p},\beta_{nj}\right) - d_{nj}\,h_2\left(t;\alpha_n,\frac{\lambda_n}{a_p},\beta_{nj}\right)\right\},
\end{aligned}$$

and when $\mu \geq \tau$ the same is given by

$$f_{\widehat{\eta}_p}(t) = h_2\left(t-\mu; \alpha_1, \frac{\lambda_1}{a_p}, \frac{1}{\nu_{n-1}}\right),$$

where

$$h_1(t;\beta) = \begin{cases} g_1(t;1,\beta) & \text{if } \beta > 0 \\ g_1(-t;1,-\beta) & \text{if } \beta < 0, \end{cases}$$

$$h_2(t;\alpha,\lambda_1,\lambda_2) = \begin{cases} g_2(t;\alpha,\lambda_1,-\lambda_2) & \text{if } \lambda_2 < 0 \\ g_1(t;\alpha,\lambda_1) & \text{if } \lambda_2 = 0 \\ h_3(t;\alpha,\lambda_1,\lambda_2) & \text{if } \lambda_2 > 0, \end{cases}$$

$$h_3(t;\alpha,\lambda_1,\lambda_2) = \begin{cases} \left(1-\sum_{k=0}^{\alpha-1} p_k^{'}\right) g_1\left(t;1,\frac{1}{\lambda_2}\right) - \sum_{k=0}^{\alpha-1} p_k^{'} g_1(t;\alpha-k,\lambda_1) & \text{if } t > 0 \\ 0 & \text{otherwise,} \end{cases}$$

with $p_k^{'} = \frac{\frac{1}{\lambda_1}}{\lambda_1-\frac{1}{\lambda_2}}\left(\frac{\lambda_1}{\lambda_1-\frac{1}{\lambda_2}}\right)^k$ and functions $g_1(\cdot)$, $g_2(\cdot)$, and $g_3(\cdot)$ as given in Theorem 2.3.2.

2.4 Confidence Intervals

In this section, we present different methods for construction of CIs for the unknown parameter θ. From Theorem 2.3.2, we can find the approximate CI for θ. However, as the PDF of $\widehat{\theta}$ is quite complicated, we also present the bootstrap CI for the scale parameter.

2.4.1 Approximate Confidence interval

From CDF of $\widehat{\theta}$, approximate CI can be found based on the assumption that $P_\theta(\widehat{\theta} \leq t)$ is a strictly decreasing function of θ, for all $t \in \mathbb{R}$. Several authors including Chen and Bhattacharya [47], Gupta and Kundu [77], Kundu and Basu [90], Childs et al.

[51], Balakrishnan et al. [36] used this method to find the CI of scale parameter. Though it is difficult to verify the assumption, an extensive numerical study supports the monotonicity assumption; for example, see Figures 2.1 and 2.2.

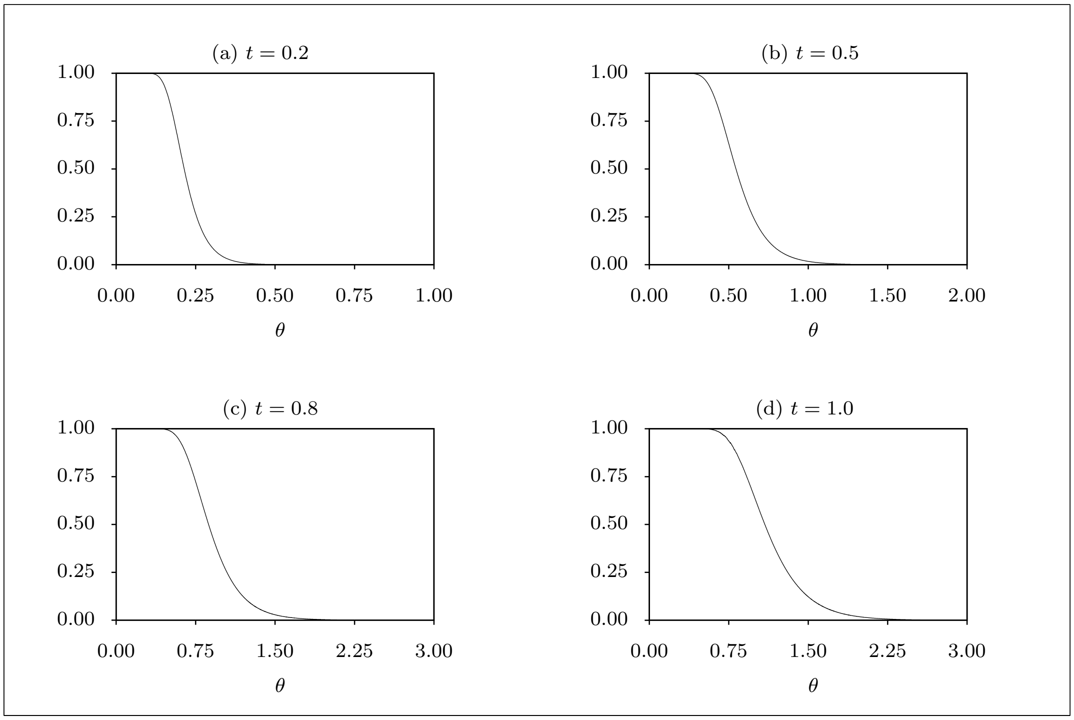


Figure 2.1: Plot of the CDF of $\widehat{\theta}$ as a function of θ for $n = 20$, $r = 16$, $\mu = 0.0$, $\tau = 0.5$ and different values of t.

Suppose $\widehat{\theta}_{\text{obs}}$ is MLE of θ. Then a two-sided $100(1-\gamma)\%$ approximate CI, say (θ_L, θ_U), for θ can be constructed by solving the equations

$$F_{\theta_L}(\widehat{\theta}_{obs}) = 1 - \frac{\gamma}{2} \quad \text{and} \quad F_{\theta_U}(\widehat{\theta}_{obs}) = \frac{\gamma}{2}$$

for θ_L(the lower bound of θ) and θ_U(the upper bound of θ) with replacing μ by its MLE. However, they are nonlinear equations and one needs to involve some numerical procedure, *e.g.*, bisection method or Newton-Raphson method, to solve them.

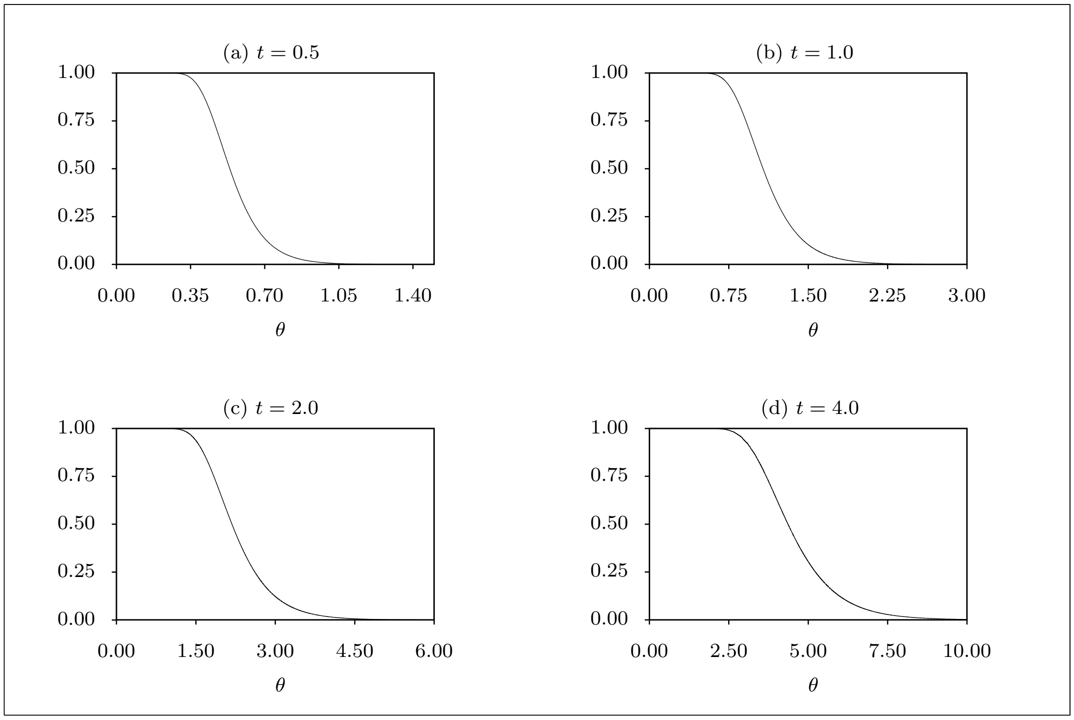


Figure 2.2: Plot of the CDF of $\widehat{\theta}$ as a function of θ for $n = 20$, $r = 16$, $\mu = 0.0$, $\tau = 2.0$ and different values of t.

2.4.2 Bootstrap Confidence Interval

The exact CIs presented in the previous section are computationally quite complicated, specially when sample size is large. So we consider the bootstrap CIs. Here we consider two types of bootstrap CI, *viz.*, percentile bootstrap CI and bias adjusted percentile (BCa) bootstrap CI; see Efron and Tibshirani [68] for details.

Bootstrap sample

Step 1. Given τ, r, n and the original Type-II sample, $\widehat{\mu}$ and $\widehat{\theta}$ are obtained from (2.3) or (2.4).

Step 2. Based on τ, r, n, $\widehat{\mu}$ and $\widehat{\theta}$, a random sample of size n is generated from Uniform(0,1) distribution and the order them to get $(U_{1:n}, \ldots, U_{n:n})$.

Step 3. Let

$$t^*_{i:n} = \widehat{\mu} - \widehat{\theta}\ log(1 - U_{i:n}).$$

Step 4. If $t^*_{r:n} < \tau$, find N_1 such that

$$t^*_{N_1:n} < \tau \leq t^*_{N_1+1:n} \quad \text{and set} \quad N^* = \begin{cases} N_1 & \text{if} \quad t^*_{r:n} < \tau \\ r & \text{if} \quad t^*_{r:n} \geq \tau. \end{cases}$$

Now, $\{t^*_{1:n}, \ldots, t^*_{N^*:n}\}$ is the bootstrap sample.

Step 5. Based on n, τ, r, and the bootstrap sample, $\widehat{\mu}^*$ and $\widehat{\theta}^*$ are obtained form (2.3) or (2.4).

Step 6. Steps 1-5 are repeated B times and $\widehat{\theta}^*$'s are ordered in ascending order to obtain the bootstrap sample

$$\left\{\widehat{\theta}^{*[1]}, \widehat{\theta}^{*[2]}, \ldots, \widehat{\theta}^{*[B]}\right\}.$$

Percentile bootstrap CI

A two-sided $100(1-\gamma)\%$ bootstrap confidence interval for θ is

$$\left(\widehat{\theta}^{*[\frac{\gamma}{2}B]}, \widehat{\theta}^{*[(1-\frac{\gamma}{2})B]}\right),$$

where, $[x]$ denotes the largest integer less than or equal to x.

Bias adjusted percentile (BCa) interval

A two-sided $100(1-\gamma)\%$ BCa bootstrap confidence interval for θ is

$$\left(\widehat{\theta}^{*[\gamma_1 B]}, \widehat{\theta}^{*[\gamma_2 B]}\right),$$

where

$$\gamma_1 = \Phi\left\{\widehat{z}_0 + \frac{\widehat{z}_0 + z_{1-\gamma/2}}{1 - a(\widehat{z}_0 + z_{1-\gamma/2})}\right\} \qquad \text{and} \qquad \gamma_2 = \Phi\left\{\widehat{z}_0 + \frac{\widehat{z}_0 + z_{\gamma/2}}{1 - a(\widehat{z}_0 + z_{\gamma/2})}\right\}.$$

Here $\Phi(\cdot)$ is the CDF of the standard normal distribution, z_γ is the upper γ-point of standard normal distribution, and

$$\widehat{z}_0 = \Phi^{-1}\left\{\frac{\#\text{ of } \widehat{\theta}^{*[j]} < \widehat{\theta}}{B}\right\}, \quad j = 1, \dots, B.$$

A estimate of the acceleration a is

$$\widehat{a} = \frac{\sum_{i=1}^{N^*}\left[\widehat{\theta}^{(\cdot)} - \widehat{\theta}^{(i)}\right]^3}{6\left\{\sum_{i=1}^{N^*}\left[\widehat{\theta}^{(\cdot)} - \widehat{\theta}^{(i)}\right]^2\right\}^{3/2}},$$

where $\widehat{\theta}^{(i)}$ is the MLE of θ based on the original sample with the ith observation deleted, and

$$\widehat{\theta}^{(\cdot)} = \frac{1}{N^*}\sum_{i=1}^{N^*}\widehat{\theta}^{(i)}.$$

2.5 Approximation for Distribution of Maximum Likelihood Estimator

Since the exact distribution of $\widehat{\theta}$ and the associated confidence interval become very complicated, it seems meaningful if this distribution can be approximated by some other well known distribution. We see that the distribution of $\widehat{\theta}$ is very close to the gamma distribution. Note that, if $\tau < \mu$ then the distribution of $\widehat{\theta}$ has a gamma distribution. Again if $(\tau - \mu)/\theta$ is very large, then the distribution of $\widehat{\theta}$ will also become very close to gamma distribution. Hence, we have tried to approximate the distribution of $\widehat{\theta}$ by a gamma distribution. The gamma parameters are found by equating the first two moments of the distribution of $\widehat{\theta}$ and that of a gamma distribution. In Figure 2.3, some plots of exact PDF of $\widehat{\theta}$ and its gamma approximation with histogram are given with $\mu = 0$, $\theta = 1$, $n = 20$, $r = 16$, and for different values of τ. These histograms are drawn with $10,000$ replications. The plots are indicating that these approximations are quite good. In Table 2.1, values of gamma parameters are reported for the above mentioned parameters. CIs for θ can be found using the gamma approximation exactly the same way as describes in Section 2.4.

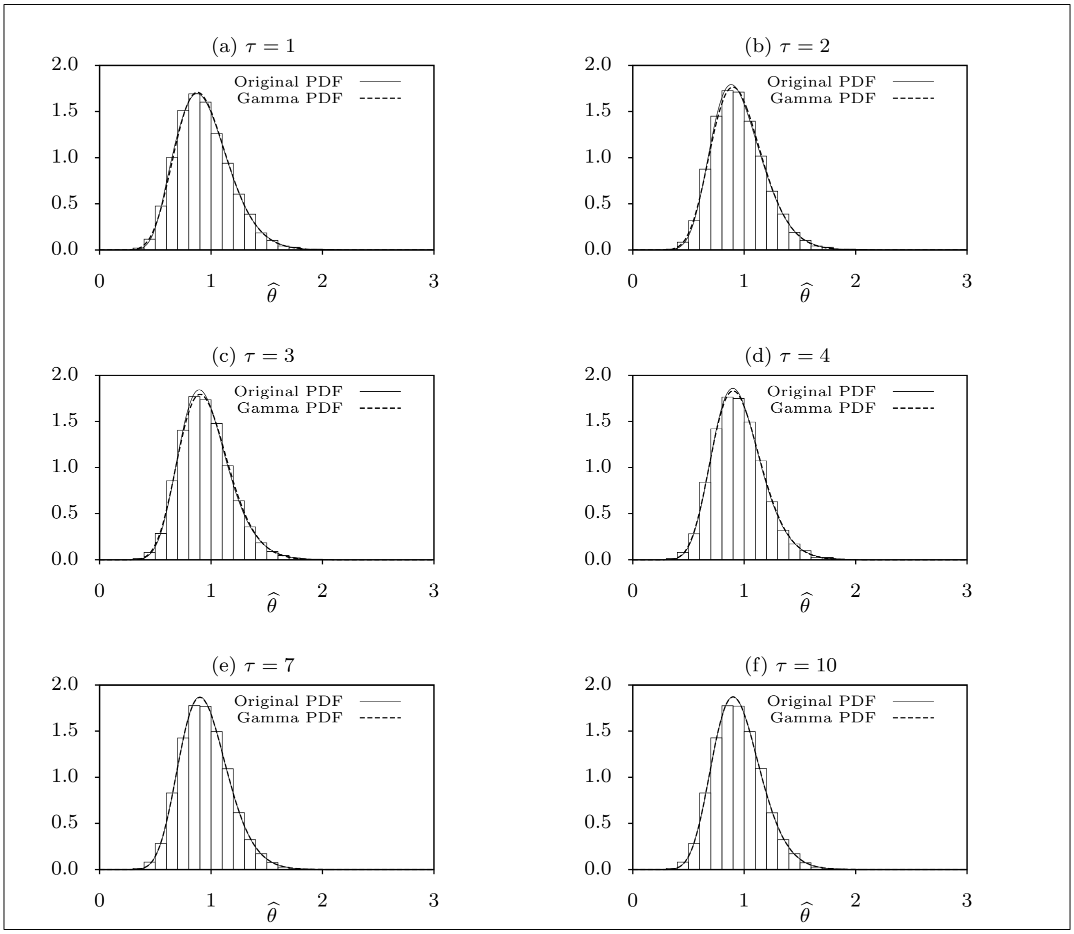


Figure 2.3: For $n = 20$, $r = 16$, $\mu = 0$, $\theta = 1$ and different values of τ plot of histogram, exact PDF of $\widehat{\theta}$ and Gamma PDF.

Table 2.1: Gamma parameters for $\mu = 0$, $\theta = 1$, $n = 20$, and different values of τ

(a) $r = 12$

τ	α	λ
0.25	11.00	12.00
0.50	11.20	12.20
0.75	12.44	13.40
1.00	13.77	14.56
1.50	14.48	15.06
2.00	15.71	16.35
2.50	16.80	17.54
3.00	17.51	18.32
3.50	17.97	18.83
4.00	18.28	19.18
7.00	18.91	19.90
10.00	18.99	19.99

(b) $r = 16$

τ	α	λ
0.25	15.00	16.00
0.50	15.00	16.00
0.75	15.04	16.04
1.00	15.29	16.29
1.50	16.52	17.44
2.00	17.01	17.80
2.50	17.20	17.97
3.00	17.59	18.41
3.50	17.99	18.85
4.00	18.28	19.18
7.00	18.91	19.90
10.00	18.99	19.99

Note that the distribution of $\widehat{\theta}/\theta$ depends on τ, μ, and θ only through $(\tau-\mu)/\theta$. Hence, Table 2.1 can be used for other values of τ, μ and θ. To use this table for the other values of τ, μ, and θ, say $\tau=\tau_1$, $\mu=\mu_1$, and $\theta=\theta_1$, one needs to compute the value of $(\tau_1-\mu_1)/\theta_1$ and takes those gamma parameters which corresponds to the $\tau=(\tau_1-\mu_1)/\theta_1$ in Table 2.1.

2.6 Simulation Study

In this section the results of Monte Carlo simulation are presented to study the performance of the inference procedures described in Sections 2.2, 2.4, and 2.5. We choose the value of the location parameter μ to be zero (without loss of generality) and different values for the scale parameter θ, *viz.*, 0.50, 1.00, 2.00, 3.00, 4.00, and 5.00. We also take $n=20$, $r=16$ and different choices for τ. Note that

$$E(\widehat{\mu})=\mu+\frac{\theta}{n}.$$

Depending on this relation one of the examiner had suggested to judge the performance of the following bias-reduced estimator of μ,

$$\widetilde{\mu}=\widehat{\mu}-\frac{\widehat{\theta}}{n}.$$

The AEs and MSEs of $\widehat{\mu}$ and $\widetilde{\mu}$ as estimators of μ and those of $\widehat{\theta}$ are reported in the Table 2.2. The coverage percentage of different CIs discussed in the section 2.4 are calculated based on the $10,000$ Monte Carlo simulations and $B=1000$. These values are presented in the Tables 2.3 and 2.4. All the generations of random numbers in this dissertation have been based on the 'ran2' function in Press et al. [123]. The value of the seed has been taken to be one.

It is clear form the Table 2.2 that the performance of $\widetilde{\mu}$ is much better than that of $\widehat{\mu}$. Table 2.3 reveals that the approximate method of constructing CI is always maintaining its CP to its pre-fixed nominal level. Among the bootstrap methods for constructing confidence interval, adjusted percentile bootstrap method is better

Table 2.2: Average estimates and MSEs of MLE and bias-reduced estimator of μ.

		$\widehat{\mu}$		$\widetilde{\mu}$		$\widehat{\theta}$	
τ	θ	AE	MSE	AE	MSE	AE	MSE
0.50	0.50	0.025	0.0013	0.002	0.0007	0.467	0.0156
	1.50	0.025	0.0013	0.001	0.0007	0.934	0.0634
	2.50	0.025	0.0013	0.002	0.0007	1.867	0.2536
	3.50	0.025	0.0013	0.002	0.0007	2.801	0.5705
	4.50	0.025	0.0013	0.002	0.0007	3.735	1.0143
1.00	0.50	0.051	0.0051	0.004	0.0027	4.668	1.5848
	1.50	0.051	0.0051	0.003	0.0027	0.476	0.0136
	2.50	0.051	0.0051	0.003	0.0027	0.943	0.0582
	3.50	0.051	0.0051	0.003	0.0027	1.868	0.2529
	4.50	0.051	0.0051	0.003	0.0027	2.801	0.5705
2.00	0.50	0.101	0.0206	0.008	0.0110	3.735	1.0143
	1.50	0.101	0.0206	0.008	0.0110	4.668	1.5848
	2.50	0.101	0.0206	0.007	0.0110	0.474	0.0128
	3.50	0.101	0.0206	0.006	0.0109	0.953	0.0559
	4.50	0.101	0.0206	0.006	0.0109	1.876	0.2413
3.00	0.50	0.152	0.0463	0.012	0.0247	2.802	0.5679
	1.50	0.152	0.0463	0.012	0.0247	3.735	1.0136
	2.50	0.152	0.0463	0.012	0.0247	4.668	1.5847
	3.50	0.152	0.0463	0.011	0.0247	0.474	0.0126
	4.50	0.152	0.0463	0.010	0.0247	0.952	0.0533
4.00	0.50	0.202	0.0823	0.015	0.0439	1.897	0.2284
	1.50	0.202	0.0823	0.015	0.0439	2.810	0.5494
	2.50	0.202	0.0823	0.015	0.0439	3.736	1.0074
	3.50	0.202	0.0823	0.015	0.0439	4.669	1.5828
	4.50	0.202	0.0823	0.015	0.0439	0.474	0.0126
5.00	0.50	0.253	0.1286	0.019	0.0686	0.950	0.0519
	1.50	0.253	0.1286	0.019	0.0686	1.906	0.2254
	2.50	0.253	0.1286	0.019	0.0686	2.830	0.5237
	3.50	0.253	0.1286	0.019	0.0686	3.745	0.9823
	4.50	0.253	0.1286	0.019	0.0686	4.671	1.5725

then percentile bootstrap method with respect to the CP. From Table 2.4, we can see that the CP of the percentile bootstrap method is quite lower than its pre-fixed nominal level, while the same of the adjusted bootstrap method is somewhat close to its nominal level.

Next we consider confidence intervals under the gamma approximation to the distribution of $\widehat{\theta}$. Simulation results for $\mu = 0$, $\theta = 1$, $n = 20$, $r = 16$ and for different values of τ are given in Table 2.5. It can be seen that the CP is very close to nominal level under this approximation. Also we see that the AL of these CIs

Table 2.3: CP and AL of approximate CIs based on 10000 simulations with $\mu = 0$, $n = 20$, and $r = 16$.

		90% C.I.		95% C.I.		99% C.I.	
τ	θ	CP	AL	CP	AL	CP	AL
0.50	0.50	90.17	0.47	95.48	0.57	99.27	0.81
	1.00	89.65	0.93	95.39	1.15	99.15	1.62
	2.00	89.18	1.80	94.61	2.32	98.01	3.23
	3.00	86.88	2.67	93.24	3.49	97.34	4.75
	4.00	87.87	3.59	94.53	4.61	98.34	6.27
	5.00	87.76	4.55	94.32	5.72	98.30	7.82
1.50	1.00	89.98	0.93	95.35	1.14	99.27	1.60
	1.50	90.17	1.40	95.48	1.72	99.27	2.42
	2.00	89.87	1.87	95.44	2.29	99.26	3.23
	3.00	89.63	2.79	95.38	3.44	99.16	4.85
	4.00	89.18	3.69	95.15	4.59	98.96	6.48
	5.00	88.51	4.55	94.63	5.77	98.51	8.10
2.50	1.00	89.66	0.87	95.03	1.07	99.20	1.49
	2.00	90.13	1.87	95.46	2.29	99.26	3.22
	2.50	90.17	2.34	95.48	2.86	99.27	4.03
	3.00	90.00	2.80	95.43	3.44	99.28	4.84
	4.00	90.00	3.73	95.40	4.59	99.16	6.46
	5.00	89.60	4.65	95.38	5.73	99.14	8.08
3.50	1.00	89.88	0.83	95.11	1.02	99.02	1.42
	2.00	89.85	1.84	95.22	2.25	99.21	3.16
	3.00	90.18	2.80	95.51	3.44	99.22	4.84
	3.50	90.17	3.27	95.48	4.01	99.27	5.65
	4.00	90.01	3.74	95.36	4.58	99.27	6.46
	5.00	89.83	4.67	95.28	5.73	99.23	8.07
4.50	1.00	89.72	0.82	95.14	1.00	99.13	1.39
	2.00	89.78	1.77	95.25	2.17	99.19	3.04
	3.00	89.98	2.78	95.35	3.41	99.27	4.80
	4.00	90.34	3.74	95.45	4.58	99.26	6.45
	4.50	90.17	4.21	95.48	5.16	99.27	7.26
	5.00	89.94	4.67	95.43	5.73	99.25	8.07

are very close to that of CIs calculated from the exact distribution of $\widehat{\theta}$. Though approximate method of construction of confidence interval is always better with respect to CP and AL, it is very complicated to calculate specially when n is large. So we suggest to use the BCa bootstrap CI or gamma approximation CI for the large values of n.

2.7 Data Analysis

In this section we consider a data set to illustrate the procedures described in the previous sections. Here we consider the data provided by Bain and Englehardt [8]. A sample of 20 items had been put on the test and the test had been terminated

Table 2.4: CP and AL of bootstrap CI based on 10000 simulations with $\mu = 0$, $n = 20$, $r = 16$, and $B = 1000$.

		Bootstrap CI						BCa Bootstrap CI					
		90% C.I.		95% C.I.		99% C.I.		90% C.I.		95% C.I.		99% C.I.	
τ	θ	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL
0.50	0.50	78.34	0.36	85.07	0.43	93.39	0.56	86.09	0.43	92.75	0.52	97.23	0.62
	1.00	78.28	0.74	84.78	0.88	92.84	1.14	85.76	0.88	92.18	1.04	96.57	1.24
	2.00	78.28	1.47	84.78	1.75	92.84	2.29	85.86	1.77	92.12	2.09	96.49	2.48
	3.00	78.28	2.21	84.78	2.63	92.84	3.44	85.70	2.67	92.05	3.15	96.39	3.71
	4.00	78.28	2.94	84.78	3.50	92.84	4.58	85.57	3.58	91.92	4.23	96.18	4.92
	5.00	78.28	3.68	84.78	4.38	92.84	5.73	85.54	4.50	91.72	5.33	95.73	6.09
1.50	0.50	81.15	0.35	87.62	0.42	94.91	0.56	85.39	0.43	91.67	0.51	96.87	0.61
	1.00	81.47	0.72	88.02	0.85	95.00	1.12	87.61	0.87	93.67	1.03	97.81	1.23
	2.00	78.28	1.47	84.78	1.74	92.83	2.27	85.85	1.77	92.15	2.09	96.56	2.48
	3.00	78.28	2.21	84.78	2.63	92.84	3.43	85.70	2.67	92.05	3.15	96.39	3.71
	4.00	78.28	2.94	84.78	3.50	92.84	4.58	85.57	3.58	91.92	4.23	96.18	4.92
	5.00	78.28	3.68	84.78	4.38	92.84	5.73	85.54	4.50	91.72	5.33	95.73	6.09
2.50	0.50	80.67	0.34	87.31	0.41	94.67	0.53	85.65	0.41	91.93	0.48	96.82	0.58
	1.00	81.93	0.71	88.26	0.85	95.04	1.12	85.36	0.87	91.70	1.03	97.14	1.23
	2.00	79.68	1.44	86.53	1.71	94.39	2.24	87.03	1.76	93.22	2.08	97.47	2.46
	3.00	78.28	2.20	84.78	2.61	92.85	3.40	85.72	2.67	92.10	3.15	96.57	3.70
	4.00	78.28	2.94	84.78	3.50	92.84	4.57	85.58	3.58	91.93	4.23	96.20	4.92
	5.00	78.28	3.68	84.78	4.38	92.84	5.72	85.54	4.50	91.72	5.33	95.73	6.09
3.50	0.50	80.34	0.34	87.02	0.40	94.06	0.51	85.67	0.40	92.10	0.47	96.90	0.57
	1.00	80.87	0.70	87.44	0.83	94.56	1.10	85.44	0.85	91.60	1.00	96.91	1.21
	2.00	82.47	1.43	88.70	1.71	95.39	2.24	87.20	1.76	93.26	2.07	97.69	2.46
	3.00	79.05	2.17	86.02	2.58	94.04	3.36	86.74	2.66	93.03	3.14	97.24	3.68
	4.00	78.28	2.92	84.78	3.47	92.96	4.53	85.60	3.57	92.06	4.23	96.45	4.91
	5.00	78.28	3.67	84.78	4.36	92.84	5.70	85.55	4.50	91.73	5.33	95.76	6.08
4.50	0.50	80.69	0.34	87.24	0.40	94.59	0.52	85.62	0.40	92.03	0.47	96.97	0.56
	1.00	80.59	0.69	87.14	0.82	94.19	1.08	85.55	0.83	91.79	0.98	96.75	1.18
	2.00	81.99	1.43	88.40	1.71	95.13	2.25	85.65	1.76	92.11	2.07	97.31	2.46
	3.00	81.47	2.15	88.02	2.56	95.00	3.36	87.45	2.66	93.49	3.14	97.61	3.68
	4.00	78.79	2.89	85.60	3.44	93.86	4.49	86.48	3.57	92.78	4.22	96.97	4.89
	5.00	78.28	3.65	84.81	4.33	93.03	5.65	85.60	4.50	91.95	5.32	96.09	6.07

after 150 hours. There were 13 failures within first 150 hours of the test, and the failures times were 3, 19, 23, 26, 37, 38, 41, 45, 58, 84, 90, 109, and 138. One can choose any r less than or equal to 13 to transform it a hybrid Type-II censored data. Here we take $r = 12$. Under the assumption of two-parameters exponential distributed lifetimes, MLEs of μ and θ are 3 and 130.85, respectively. Different types of confidence interval are reported in the Table 2.6.

Next we consider the gamma approximation to the original PDF of $\widehat{\theta}$. Here we have $\tau = 150$, $\widehat{\mu} = 3$, and $\widehat{\theta} = 130.85$. Hence, $(\tau - \widehat{\mu})/\widehat{\theta} = 1.12$ and the corresponding values of shape and scale parameters of gamma approximation are 14.04 and 14.73,

Table 2.5: CP and AL of CI obtain from gamma approximation based on 10000 simulations with $\mu = 0$, $\theta = 1$, $n = 20$, and $r = 16$.

		90% C.I.		95% C.I.		99% C.I.	
r	τ	CP	AL	CP	AL	CP	AL
12	1.00	91.00	1.13	95.88	1.41	99.12	2.06
	1.50	90.68	1.06	95.37	1.34	99.10	2.01
	2.50	90.29	0.93	95.35	1.16	99.15	1.72
	3.50	90.15	0.88	95.21	1.09	98.98	1.59
	4.50	89.61	0.85	95.04	1.05	98.79	1.52
16	1.00	90.36	0.93	95.63	1.14	99.37	1.61
	1.50	90.32	0.93	95.57	1.15	99.24	1.62
	2.50	90.06	0.89	95.16	1.10	99.18	1.58
	3.50	90.13	0.86	95.21	1.05	98.98	1.50
	4.50	89.94	0.84	95.16	1.03	99.11	1.46

Table 2.6: Lower and upper limits of different confidence intervals for θ

	Approx. CI		Per Boot. CI		BCa Boot. CI		Gam. Approx. CI	
Level	LL	UL	LL	UL	LL	UL	LL	UL
90%	91.75	241.02	75.61	185.33	102.27	182.76	82.06	200.13
95%	84.97	270.12	70.57	199.49	95.63	182.76	76.30	221.27
99%	73.57	342.61	60.82	241.91	85.64	182.76	66.53	271.71

respectively. Using the Gamma(14.04, 14.73) distribution, the CI is reported in Table 2.6.

2.8 Conclusion

In this chapter, we have considered the HCS-II, when lifetimes are assumed to have two-parameter exponential distribution. We have found the MLEs for both the parameters. We have considered different methods for construction of CI. We have seen that the approximate and BCa bootstrap methods for construction of CI are quite good. Also we approximate the distribution of the MLE of the scale parameter by gamma distribution and find the associated CI. We have seen that this method of constructing CI is also quite good. we recommend to use the gamma approximation or BCa bootstrap method for constructing CI specially when n is large.

2.A Lemmas

We need the following lemmas to prove the Theorems 2.3.1, 2.3.2, and 2.3.3.

Lemma 2.A.1. Let $T_{1:n} < \ldots < T_{n:n}$ be the order statistics of a random sample of size n from a continuous distribution with PDF $f(x)$ and the corresponding CDF is $F(\cdot)$. Let τ be a pre-fixed number such that $F(\tau) > 0$, and N denote the number of order statistics less than or equal to τ. The conditional joint PDF of $T_{1:n}, \ldots, T_{N:n}$ conditioned on the event $N = i$, $i = 1, 2, \cdots, n$, is given by

$$f\left(t_1, t_2, \ldots, t_N \,|\, N = i\right) = \frac{n!}{(n-i)!P[N=i]} \prod_{j=1}^{i} f\left(t_j\right) \left\{1 - F\left(T\right)\right\}^{n-i}, \quad \text{if } t_1 < \ldots < t_i < \tau.$$

Proof: See Childs et al. [51].

Lemma 2.A.2. Let $T_{1:n} < \ldots < T_{n:n}$ be the order statistics of a random sample of size n from a continuous distribution with PDF $f(x)$ and the corresponding CDF is $F(\cdot)$. Let τ be a pre-fixed number such that $F(\tau) > 0$, and N denote the number of order statistics less than or equal to τ. Let $r \in \{1, 2, \ldots, n\}$ be a pre-fixed integer. Then PDF of $T_{1:n}, T_{2:n}, \ldots, T_{r:n}$ conditioned on the event $N = 0$ is given by

$$f\left(t_1, t_2, \ldots, t_r \,|\, N = 0\right) = \frac{n!}{(n-r)!P[N=0]} \prod_{j=1}^{r} f\left(t_j\right) \left\{1 - F\left(t_r\right)\right\}^{n-r}, \quad \text{if } \tau < t_1 < \ldots < t_r < \infty.$$

For $i = 1, 2, \ldots, r-1$, PDF of $T_{1:n}, T_{2:n}, \ldots, T_{r:n}$ conditioned on the event $N = i$ is given by

$$f\left(t_1, t_2, \ldots, t_r \,|\, N = i\right) = \frac{n!}{(n-r)!P[N=i]} \prod_{j=1}^{r} f\left(t_j\right) \left\{1 - F\left(t_r\right)\right\}^{n-r}, \quad \text{if } t_1 < \ldots < t_i < \tau < t_{i+1} < \ldots < t_r < \infty.$$

Proof: See Childs et al. [51].

Lemma 2.A.3. Let X be a Gamma(α, 1) random variable and Y be a standard Exponential random variable and they are independently distributed. The PDF of

X is

$$f_{\text{Gamma}}(x;\alpha,\lambda) = \begin{cases} \frac{1}{\lambda^{\alpha}\,\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-x/\lambda} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

with $\lambda = 1$. The PDF of Y is

$$f_2(y) = \begin{cases} e^{-y} & \text{for } y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then for any arbitrary constant A, λ_1 and λ_2 the MGF of $A + \lambda_1 X + \lambda_2 Y$ is given by

$$M_{A+\lambda_1 X + \lambda_2 Y}(\omega) = e^{\omega A}\,(1 - \lambda_1\,\omega)^{-\alpha}\,(1 - \lambda_2\,\omega)^{-1}.$$

This MGF exists if

$$\omega \quad \in \quad \begin{cases} (\frac{1}{\lambda_2}, \frac{1}{\lambda_1}) & \text{if } \lambda_2 < 0 \\ (-\infty, \min\{\frac{1}{\lambda_1}, \frac{1}{\lambda 2}\}) & \text{if } \lambda_2 \geq 0 \end{cases}$$

Proof: This lemma can be proved using the joint distribution of (X, Y), and therefore the proof is omitted.

Lemma 2.A.4. Let X be a Gamma(α, λ_1)(with α integer) and Y be an Exponential random variable with mean $\dfrac{1}{\lambda_2}$ and they be independently distributed. Then the PDF of $X - Y$ is given by

$$g_2\,(t;\alpha,\lambda_1,\lambda_2) = \sum_{k=0}^{\alpha-1} p_k\, g_1\,(t;\alpha - k,\lambda_1) + \left(1 - \sum_{k=0}^{\alpha-1} p_k\right)\, g_1\,(-t, 1, \lambda_2) \quad \text{for } t \in \mathbb{R},$$

where $p_k = \frac{\lambda_2}{\lambda_1+\lambda_2}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k$.

Proof: See Childs et al. [49].

Lemma 2.A.5. Let X be a Gamma(α, λ_1)(with α integer) and Y be a Exponential random variable with mean λ_2 and they be independently distributed. Then the PDF of $X + Y$ is given by

$$g_3\,(t;\alpha,\lambda - 1,\lambda_2) = \left(1 - \sum_{k=0}^{\alpha-1} p_k\right)\, g_1\left(t; 1, \frac{1}{\lambda_2}\right) - \sum_{k=0}^{\alpha-1} p_k\, g_1(t;\alpha - k,\lambda_1) I_{(0,\infty)}(t)$$

with $p_k = \frac{\frac{1}{\lambda_2}}{\lambda_1 - \frac{1}{\lambda_2}} \left(\frac{\lambda_1}{\lambda_1 - \frac{1}{\lambda_2}} \right)^k$.

2.B Proof of Theorems 2.3.1, 2.3.2, and 2.3.3

The number of unit failed before time τ, N, is a non-negative random variable with the PMF

$$P[N = i] = \binom{n}{i} \left(1 - e^{-\frac{\tau-\mu}{\theta}}\right)^i e^{-(n-i)\frac{\tau-\mu}{\theta}} \quad \text{if } i = 0,\, 1,\, \ldots,\, n. \tag{2.10}$$

Case-II: $r = 1$

The joint conditional MGF of $\widehat{\theta}$ and $\widehat{\mu}$ conditioned on the event $\{N \geq 1\}$ can be written as

$$E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}} \,|\, N \geq 1] \;=\; \sum_{i=1}^{n} E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}} \,|\, N = i] \times P[N = i \,|\, N \geq 1]. \tag{2.11}$$

Using (2.3) and Lemma 2.A.1,

$$\begin{aligned} E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}} \,|\, N = 1] &= \frac{1}{\theta\left(1 - e^{-\frac{\tau-\mu}{\theta}}\right)} \int_{\mu}^{\tau} e^{\omega_1 (n-1)(\tau - t) + \omega_2 t - \frac{1}{\theta}(t-\mu)} dt \\ &= \frac{e^{(n-1)\tau\omega_1 + \frac{\mu}{\theta}} \left\{ e^{-(n-1)\mu\,\omega_1 + \mu\,\omega_2 - \frac{\mu}{\theta}} - e^{-\frac{\tau}{\theta} - (n-1)\tau\omega_1 + \tau\omega_2} \right\}}{\theta\left(1 - e^{-\frac{\tau-\mu}{\theta}}\right)\left(\frac{1}{\theta} + (n-1)\,\omega_1 - \omega_2\right)}. \end{aligned}$$

Using the above expression and (2.10)

$$\begin{aligned} &E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}} \,|\, N = 1] \times P[N = 1 \,|\, N \geq 1] \\ &= (1 - q^n)^{-1} \left[c_{10} \frac{e^{\mu_{10}\,\omega_1 + \mu\,\omega_2}}{(1 + (n-1)\,\theta\omega_1 - \theta\omega_2)} - d_{10} \frac{e^{\tau\omega_2}}{(1 + (n-1)\,\theta\omega_1 - \theta\omega_2)} \right], \end{aligned} \tag{2.12}$$

where q, c_{10}, d_{10} and μ_{10} are as in Theorem 2.3.1.

Let us define for $s = 2, \cdots, n$ and for some constant a, b

$$I_s\,(a, b) = \int_{\mu}^{\tau}\!\!\int_{t_1}^{\tau}\!\!\cdots\!\int_{t_{s-1}}^{\tau} e^{-a(t_1 - \tau) - b\sum_{i=2}^{s}(t_i - \tau)} dt_s dt_{s-1} \ldots dt_1$$

$$= \int_{\mu}^{\tau}\int_{t_1}^{\tau}\cdots\int_{t_{s-2}}^{\tau} e^{-a(t_1-\tau)-b\sum_{i=2}^{s-1}(t_i-\tau)} \times \left\{\frac{e^{-b(t_{s-1}-\tau)}-1}{b}\right\} dt_{s-1}\dots dt_2 dt_1$$

$$= \frac{1}{2b^2}\int_{\mu}^{\tau}\int_{t_1}^{\tau}\cdots\int_{t_{s-3}}^{\tau} e^{-a(t_1-\tau)-b\sum_{i=2}^{s-2}(t_i-\tau)} \left\{e^{-b(t_{s-2}-\tau)-1}\right\}^2 dt_{s-2}\dots dt_2 dt_1$$

$$\vdots$$

$$= \frac{b^{-(s-1)}}{(s-1)!}\int_{\mu}^{\tau} e^{-a(t_1-\tau)}\left\{e^{-b(t_1-\tau)}-1\right\}^{s-1} dt_1$$

$$= \frac{b^{-(s-1)}}{(s-1)!}\sum_{j=0}^{s-1}(-1)^{s-j-1}\binom{s-1}{j}\int_{\mu}^{\tau} e^{-(a+bj)(t_1-\tau)}dt_1$$

$$= \frac{b^{-(s-1)}}{(s-1)!}\sum_{j=0}^{s-1}(-1)^{s-j-1}\binom{s-1}{j}\times\frac{e^{(a+bj)(\tau-\mu)}-1}{a+bj}. \qquad (2.13)$$

For $i = 2, 3, \dots, n$, using (2.3) and Lemma 2.A.1 again, we have

$$E[e^{\omega_1\widehat{\theta}+\omega_2\widehat{\mu}} \mid N=i]$$
$$= \frac{i!}{\theta^i\left(1-e^{-\frac{\tau-\mu}{\theta}}\right)^i}$$
$$\times\int_{\mu}^{\tau}\int_{t_1}^{\tau}\cdots\int_{t_{i-1}}^{\tau} e^{\frac{\omega_1}{i}\left\{\sum_{j=2}^{i}t_j-(n-1)t_1+(n-i)\tau\right\}+\omega_2 t_1-\frac{1}{\theta}\sum_{j=1}^{i}(t_j-\mu)}dt_i\dots dt_1.$$

Consider the exponential term in the integrand

$$\frac{\omega_1}{i}\left\{\sum_{j=2}^{i}t_j-(n-1)\,t_1+(n-i)\,\tau\right\}+\omega_2 t_1-\frac{1}{\theta}\sum_{j=1}^{i}(t_j-\mu)$$

$$= -\left(\frac{1}{\theta}+\frac{n-1}{i}\omega_1-\omega_2\right)(t_1-\tau)-\left(\frac{1}{\theta}-\frac{\omega_1}{i}\right)\sum_{j=2}^{i}(t_j-\tau)$$
$$-\left(\frac{1}{\theta}+\frac{n-1}{i}\omega_1-\omega_2\right)\tau-\left(\frac{1}{\theta}-\frac{\omega_1}{i}\right)(i-1)\,\tau+\frac{i\mu}{\theta}+\frac{\omega_1}{i}(n-i)\,\tau$$

$$= -\left(\frac{1}{\theta}+\frac{n-1}{i}\omega_1-\omega_2\right)(t_1-\tau)-\left(\frac{1}{\theta}-\frac{\omega_1}{i}\right)\sum_{j=2}^{i}(t_j-\tau)-i\left(\frac{\tau-\mu}{\theta}\right)+\tau\omega_2.$$

Hence, using (2.13)

$$
\begin{aligned}
&E[e^{\omega_1 \widehat{\theta}+\omega_2 \mu} \,|\, N=i] \\
&= \frac{i!\, e^{-i\,\frac{\tau-\mu}{\theta}+\tau\omega_2}}{\theta^i\left(1-e^{-\frac{\tau-\mu}{\theta}}\right)^i} \int_\mu^\tau\!\!\int_{t_1}^\tau\!\!\cdots\!\int_{t_{i-1}}^\tau e^{-\left(\frac{1}{\theta}+\frac{n-1}{i}\omega_1-\omega_2\right)(t_1-\tau)-\left(\frac{1}{\theta}-\frac{\omega_1}{i}\right)\sum_{j=2}^i (t_j-\tau)} dt_i \ldots dt_1 \\
&= \frac{e^{-i\,\frac{\tau-\mu}{\theta}+\tau\omega_2}}{\left(1-e^{-\frac{\tau-\mu}{\theta}}\right)^i} \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{i}{j+1} \times \frac{e^{\left(\frac{j+1}{\theta}+\frac{n-j-1}{\theta}\omega_1-\omega_2\right)(\tau-\mu)}-1}{\left(1-\frac{\theta}{i}\omega_1\right)^{i-1}\left(1+\frac{(n-j-1)\theta}{i(j+1)}\omega_1-\frac{\theta}{j+1}\omega_2\right)}.
\end{aligned}
$$

Using the above expression and (2.10), one will get for $i=2,\cdots,n$

$$
\begin{aligned}
&E[e^{\omega_1 \widehat{\theta}+\omega_2 \widehat{\mu}} \,|\, N=i] \times P[N=i \,|\, N\geq 1] \\
&= \frac{e^{-i\,\frac{\tau-\mu}{\theta}+\tau\omega_2}}{\left(1-e^{-\frac{\tau-\mu}{\theta}}\right)^i} \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{i}{j+1} \times \frac{e^{\left(\frac{j+1}{\theta}+\frac{n-j-1}{\theta}\omega_1-\omega_2\right)(\tau-\mu)}-1}{\left(1-\frac{\theta}{i}\omega_1\right)^{i-1}\left(1+\frac{(n-j-1)\theta}{i(j+1)}\omega_1-\frac{\theta}{j+1}\omega_2\right)} \\
&\qquad\qquad \times \frac{\binom{n}{i}\left(1-e^{-\frac{\tau-\mu}{\theta}}\right)^i e^{-(n-i)\frac{\tau-\mu}{\theta}}}{1-e^{-n\frac{\tau-\mu}{\theta}}} \\
&= \frac{e^{-n\frac{\tau-\mu}{\theta}+\tau\omega_2}}{1-e^{-n\frac{\tau-\mu}{\theta}}} \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{n}{i}\binom{i}{j+1} \frac{e^{\left(\frac{j+1}{\theta}+\frac{n-j-1}{i}\omega_1-\omega_2\right)(\tau-\mu)}-1}{\left(1-\frac{\theta}{i}\omega_1\right)^{i-1}\left(1+\frac{(n-j-1)\theta}{i(j+1)}\omega_1-\frac{\theta}{j+1}\omega_2\right)}.
\end{aligned}
$$

Hence, using (2.11), (2.12) and the above expression, we have

$$
\begin{aligned}
&E[e^{\omega_1\widehat{\theta}+\omega_2\widehat{\mu}} \,|\, N\geq 1] \\
&= (1-q^n)^{-1}\Bigg[c_{10}\,\frac{e^{\mu_{10}\omega_1+\mu\,\omega_2}}{\left(1+\frac{\omega_1}{\lambda_{10}}-\frac{\omega_2}{\nu_0}\right)} - d_{10}\,\frac{e^{\tau\omega_2}}{\left(1+\frac{\omega_1}{\lambda_{10}}-\frac{\omega_2}{\nu_0}\right)} + \frac{e^{\mu\,\omega_2}-q^n e^{\tau\omega_2}}{\left(1-\frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1-\frac{\omega_2}{\nu_{n-1}}\right)} \\
&+ \sum_{i=2}^{n-1}\sum_{j=0}^{i-1}\left\{c_{ij}\,\frac{e^{\mu_{ij}\omega_1+\mu\,\omega_2}}{\left(1-\frac{\omega_1}{\lambda_i}\right)^{\alpha_i}\left(1+\frac{\omega_1}{\lambda_{ij}}-\frac{\omega_2}{\nu_j}\right)} - d_{ij}\,\frac{e^{\tau\omega_2}}{\left(1-\frac{\omega_1}{\lambda_i}\right)^{\alpha_i}\left(1+\frac{\omega_1}{\lambda_{ij}}-\frac{\omega_2}{\nu_j}\right)}\right\} \\
&+ \sum_{j=0}^{n-2}\left\{c_{nj}\,\frac{e^{\mu_{nj}\omega_1+\mu\,\omega_2}}{\left(1-\frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1+\frac{\omega_1}{\lambda_{nj}}-\frac{\omega_2}{\nu_j}\right)} - d_{nj}\,\frac{e^{\tau\omega_2}}{\left(1-\frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1+\frac{\omega_1}{\lambda_{nj}}-\frac{\omega_2}{\nu_j}\right)}\right\}\Bigg],
\end{aligned}
$$

where q, α_i, λ_i, μ_{ij}, λ_{ij}, c_{ij} and d_{ij} are as given in Theorem 2.3.1. Now using Lemmas 2.A.3, 2.A.4, one will get (2.8).

Case-II: $r \geq 2$

The joint MGF of $\widehat{\theta}$ and $\widehat{\mu}$ can be expressed as

$$E[e^{\omega_1 \widehat{\theta}+\omega_2 \widehat{\mu}}] = \sum_{i=0}^{n} E[e^{\omega_1 \widehat{\theta}+\omega_2 \widehat{\mu}} \mid N=i] \times P[N=i]. \tag{2.14}$$

Now using (2.4) and Lemma 2.A.2, for $\omega_1 < \frac{r}{\theta}$ and $\omega_2 < \frac{1}{\theta} + \frac{n-1}{r}\,\omega_1$

$$\begin{aligned}
&E[e^{\omega_1 \widehat{\theta}+\omega_2 \widehat{\mu}} \mid N=0] \\
&= \frac{n!}{\theta^r\,(n-r)!P[N=0]} \\
&\quad \times \int_{\max\{\tau,\,\mu\}}^{\infty}\int_{t_1}^{\infty}\cdots\int_{t_{r-1}}^{\infty} \Big\{ e^{\frac{\omega_1}{r}\left\{\sum_{j=2}^{r-1} t_j-(n-1)t_1+(n-r+1)t_r\right\}+\omega_2\,t_1} \\
&\qquad\qquad \times e^{-\sum_{j=1}^{r}\left(\frac{t_j-\mu}{\theta}\right)-(n-r)\left(\frac{t_r-\mu}{\theta}\right)} \Big\}\, dt_r \ldots dt_1 \\
&= \frac{n!\,e^{\frac{n\mu}{\theta}}}{\theta^r\,(n-r)!P[N=0]} \\
&\quad \times \int_{\max\{\tau,\,\mu\}}^{\infty}\int_{t_1}^{\infty}\cdots\int_{t_{r-1}}^{\infty} e^{-\left(\frac{1}{\theta}+\frac{n-1}{r}\omega_1-\omega_2\right)t_1-\left(\frac{1}{\theta}-\frac{\omega_1}{r}\right)\sum_{j=2}^{r-1} t_j-(n-r+1)\left(\frac{1}{\theta}-\frac{\omega_1}{r}\right)t_r} dt_r \ldots dt_1 \\
&= \frac{n!\,e^{\frac{n\mu}{\theta}}}{\theta^r\,(n-r)!P[N=0]} \\
&\quad \times \int_{\max\{\tau,\,\mu\}}^{\infty}\int_{t_1}^{\infty}\cdots\int_{t_{r-2}}^{\infty} e^{-\left(\frac{1}{\theta}+\frac{n-1}{r}\omega_1-\omega_2\right)t_1-\left(\frac{1}{\theta}-\frac{\omega_1}{r}\right)\sum_{j=2}^{r-1} t_j} \\
&\qquad\qquad \times \frac{e^{-(n-r+1)\left(\frac{1}{\theta}-\frac{\omega_1}{r}\right)t_{r-1}}}{(n-r+1)\left(\frac{1}{\theta}-\frac{\omega_1}{r}\right)} dt_{r-1}\ldots dt_1 \\
&\vdots \\
&= \frac{n\,e^{\frac{n\mu}{\theta}}}{P[N=0]\theta^r} \times \frac{1}{\left(\frac{1}{\theta}-\frac{\omega_1}{r}\right)^{r-1}} \int_{\max\{\tau,\,\mu\}}^{\infty} e^{-\left(\frac{n}{\theta}-\omega_2\right)t_1} dt_1 \\
&= \frac{n\,e^{\frac{n\mu}{\theta}}}{P[N=0]\theta^r} \times \frac{e^{-\left(\frac{n}{\theta}-\omega_2\right)\max\{\tau,\,\mu\}}}{\left(\frac{1}{\theta}-\frac{\omega_1}{r}\right)^{r-1}\left(\frac{n}{\theta}-\omega_2\right)} \\
&= \frac{e^{-n\frac{\max\{\tau,\,\mu\}-\mu}{\theta}+\omega_2\max\{\tau,\,\mu\}}}{P[N=0]\left(1-\frac{\theta}{r}\omega_1\right)^{r-1}\left(1-\frac{\theta}{n}\omega_2\right)}.
\end{aligned}$$

Hence, for $\omega_1 < \frac{r}{\theta}$ and $\omega_2 < \frac{1}{\theta} + \frac{n-1}{r}\omega_1$

$$
\begin{aligned}
&E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}} \mid N = 0] \times P[N = 0] \\
&= e^{-n\frac{\max\{\tau,\mu\}-\mu}{\theta} + \omega_2 \max\{\tau,\mu\}} \times \frac{1}{\left(1 - \frac{\theta}{r}\omega_1\right)^{r-1}\left(1 - \frac{\theta}{n}\omega_2\right)}.
\end{aligned}
\tag{2.15}
$$

Now for $i = 1, 2, \ldots, r-1$ and $\omega_1 < \frac{r}{\theta}$, using (2.4) and Lemma 2.A.2

$$
\begin{aligned}
&E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}} \mid N = i] \\
&= \frac{n!}{(n-r)! P[N = i]\theta^r} \\
&\times \int_\mu^\tau \!\!\ldots\! \int_{t_{i-1}}^\tau \int_\tau^\infty \!\!\ldots\! \int_{t_{r-1}}^\infty \Big\{ e^{\frac{\omega_1}{r}\left\{\sum_{j=2}^{r-1} t_j - (n-1)t_1 + (n-r+1)t_r\right\} + \omega_2 t_1} \\
&\qquad\qquad \times e^{-\sum_{j=1}^{r}\left(\frac{t_j-\mu}{\theta}\right) - (n-r)\left(\frac{t_r-\mu}{\theta}\right)} \Big\} \, dt_r \ldots dt_1
\end{aligned}
$$

Consider the exponential term of the integrand

$$
\begin{aligned}
&\frac{\omega_1}{r}\left\{\sum_{j=2}^{r-1} t_j - (n-1)\,t_1 + (n-r+1)\,t_r\right\} + \omega_2\, t_1 - \sum_{j=1}^{r}\left(\frac{t_j-\mu}{\theta}\right) - (n-r)\left(\frac{t_r-\mu}{\theta}\right) \\
&= -\left(\frac{1}{\theta} + \frac{n-1}{r}\omega_1 - \omega_2\right) t_1 - \left(\frac{1}{\theta} - \frac{\omega_1}{r}\right)\sum_{j=2}^{r-1} t_j - (n-r+1)\left(\frac{1}{\theta} - \frac{\omega_1}{r}\right) t_r + \frac{n\mu}{\theta} \\
&= -\left(\frac{1}{\theta} + \frac{n-1}{r}\omega_1 - \omega_2\right)(t_1 - \tau) - \left(\frac{1}{\theta} + \frac{n-1}{r}\omega_1 - \omega_2\right)\tau - \left(\frac{1}{\theta} - \frac{\omega_1}{r}\right)\sum_{j=2}^{i}(t_j - \tau) \\
&\quad - \left(\frac{1}{\theta} - \frac{\omega_1}{r}\right)(i-1)\,\tau - \left(\frac{1}{\theta} - \frac{\omega_1}{r}\right)\sum_{j=i+1}^{r-1} t_j - (n-r+1)\left(\frac{1}{\theta} - \frac{\omega_1}{r}\right) t_r + \frac{n\mu}{\theta} \\
&= -\left(\frac{1}{\theta} + \frac{n-1}{r}\omega_1 - \omega_2\right)(t_1 - \tau) - \left(\frac{1}{\theta} - \frac{\omega_1}{r}\right)\sum_{j=2}^{i}(t_j - \tau) - \left(\frac{1}{\theta} - \frac{\omega_1}{r}\right)\sum_{j=i+1}^{r-1} t_j \\
&\quad - (n-r+1)\left(\frac{1}{\theta} - \frac{\omega_1}{r}\right) t_r + \frac{n\mu}{\theta} - \frac{i\tau}{\theta} - \frac{n-i}{r}\tau\omega_1 + \omega_2\,\tau.
\end{aligned}
$$

Hence, for $i = 1, 2, \ldots, r-1$ and for $\omega_1 < \frac{r}{\theta}$,

$$
\begin{aligned}
&E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}} \mid N = i] \\
&= \frac{n!\, e^{\frac{n\mu}{\theta} - \frac{i\tau}{\theta} - \frac{n-i}{r}\tau\omega_1 + \omega_2 \tau}}{(n-r)! P[N = i]\theta^r} \\
&\quad \times \left\{ \int_\mu^\tau \int_{t_1}^\tau \!\!\ldots\! \int_{t_{i-1}}^\tau e^{-\left(\frac{1}{\theta} + \frac{n-1}{r}\omega_1 - \omega_2\right)(t_1-\tau) - \left(\frac{1}{\theta} - \frac{\omega_1}{r}\right)\sum_{j=2}^{i}(t_j-\tau)} dt_i \ldots dt_1 \right\}
\end{aligned}
\tag{2.16}
$$

$$\times \left\{ \int_{\tau}^{\infty} \int_{t_{i+1}}^{\infty} \cdots \int_{t_{r-1}}^{\infty} e^{-\left(\frac{1}{\theta} - \frac{\omega_1}{r}\right) \sum_{j=i+1}^{r-1} t_j - (n-r+1)\left(\frac{1}{\theta} - \frac{\omega_1}{r}\right) t_r} dt_r \ldots dt_{i+1} \right\}. \tag{2.17}$$

Consider the integration in the first braces in the above expression

$$\int_{\mu}^{\tau} \int_{t_1}^{\tau} \cdots \int_{t_{i-1}}^{\tau} e^{-\left(\frac{1}{\theta} + \frac{n-1}{r}\omega_1 - \omega_2\right)(t_1 - \tau) - \left(\frac{1}{\theta} - \frac{\omega_1}{r}\right) \sum_{j=2}^{i} (t_j - \tau)} dt_i \ldots dt_1$$
$$= \sum_{j=0}^{i-1} \frac{(-1)^{i-j-1} \binom{i-1}{j} \left\{ e^{\left(\frac{j+1}{\theta} + \frac{n-j-1}{r}\omega_1 - \omega_2\right)(\tau - \mu)} - 1 \right\}}{(i-1)! \left(\frac{1}{\theta} - \frac{\omega_1}{r}\right)^{i-1} \left(\frac{j+1}{\theta} + \frac{n-j-1}{r}\omega_1 - \omega_2\right)}. \tag{2.18}$$

This equality is due to (2.13). Consider the integration inside the second braces in (2.17)

$$\int_{\tau}^{\infty} \int_{t_{i+1}}^{\infty} \cdots \int_{t_{r-1}}^{\infty} e^{-\left(\frac{1}{\theta} - \frac{\omega_1}{r}\right) \sum_{j=i+1}^{r-1} t_j - (n-r+1)\left(\frac{1}{\theta} - \frac{\omega_1}{r}\right) t_r} dt_r \ldots dt_{i+1}$$
$$= \int_{\tau}^{\infty} \int_{t_{i+1}}^{\infty} \cdots \int_{t_{r-2}}^{\infty} e^{-\left(\frac{1}{\theta} - \frac{\omega_1}{r}\right) \sum_{j=i+1}^{r-1} t_j} \times \frac{e^{-(n-r+1)\left(\frac{1}{\theta} - \frac{\omega_1}{r}\right) t_{r-1}}}{(n-r+1)\left(\frac{1}{\theta} - \frac{\omega_1}{r}\right)} dt_{r-1} \ldots dt_{i+1}$$
$$\vdots$$
$$= \frac{(n-r)!\, e^{-(n-i)\left(\frac{1}{\theta} - \frac{\omega_1}{r}\right)\tau}}{(n-i)! \left(\frac{1}{\theta} - \frac{\omega_1}{r}\right)^{r-i}}.$$

Hence, using above expression and (2.18) in (2.17), one will get for $\omega_1 < \frac{r}{\theta}$

$$E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}} \mid N = i]$$
$$= \frac{n!\, e^{\frac{n\mu}{\theta} - \frac{i\tau}{\theta} - \frac{n-i}{r}\tau\omega_1 + \tau\omega_2}}{(n-r)! P[N=i] \theta^r} \times \frac{(n-r)!\, e^{-(n-i)\left(\frac{1}{\theta} - \frac{\omega_1}{r}\right)\tau}}{(n-i)! \left(\frac{1}{\theta} - \frac{\omega_1}{r}\right)^{r-i}}$$
$$\times \sum_{j=0}^{i-1} \frac{(-1)^{i-j-1} \binom{i-1}{j} \left\{ e^{\left(\frac{j+1}{\theta} + \frac{n-j-1}{r}\omega_1 - \omega_2\right)(\tau - \mu)} - 1 \right\}}{(i-1)! \left(\frac{1}{\theta} - \frac{\omega_1}{r}\right)^{i-1} \left(\frac{j+1}{\theta} + \frac{n-j-1}{r}\omega_1 - \omega_2\right)}$$

$$= \frac{e^{-n\frac{\tau-\mu}{\theta} + \tau\omega_2}}{P[N=i]} \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{n}{i} \binom{i}{j+1} \times \frac{e^{\left(\frac{j+1}{\theta} + \frac{n-j-1}{r}\omega_1 - \omega_2\right)(\tau-\mu)} - 1}{\left(1 - \frac{\theta}{r}\omega_1\right)^{r-1} \left(1 + \frac{(n-j-1)\theta}{r(j+1)}\omega_1 - \frac{\theta}{j+1}\omega_2\right)}.$$

Hence, for $i = 1, 2, \ldots, r-1$ and $\omega < \frac{r}{\theta}$,

$$E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}} \mid N = i] \times P[N=i]$$
$$= e^{-n\frac{\tau-\mu}{\theta} + \tau\omega_2} \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{n}{i} \binom{i}{j+1} \times \frac{e^{\left(\frac{j+1}{\theta} + \frac{n-j-1}{r}\omega_1 - \omega_2\right)(\tau-\mu)} - 1}{\left(1 - \frac{\theta}{r}\omega_1\right)^{r-1} \left(1 + \frac{(n-j-1)\theta}{r(j+1)}\omega_1 - \frac{\theta}{j+1}\omega_2\right)} \tag{2.19}$$

Now for $i = r, r+1, \ldots, n$, using (2.4) and Lemma 2.A.1

$$
\begin{aligned}
&E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}} \mid N = i] \\
&= \frac{n!}{(n-i)!\theta^i P[N=i]} \\
&\quad \times \int_\mu^\tau \int_{t_1}^\tau \cdots \int_{t_{i-1}}^\tau e^{\frac{\omega_1}{i}\left\{\sum_{j=2}^i t_j - (n-1)t_1 + (n-i)\tau\right\} + \omega_2 t_1 - \sum_{j=1}^i \left(\frac{t_j - \mu}{\theta}\right) - (n-i)\left(\frac{\tau-\mu}{\theta}\right)} dt_i \ldots dt_1 \\
&= \frac{n!\, e^{-n\frac{\tau-\mu}{\theta} + \tau\omega_2}}{(n-i)!\theta^i P[N=i]} \\
&\quad \times \int_\mu^\tau \int_{t_1}^\tau \cdots \int_{t_{i-1}}^\tau e^{-\left(\frac{1}{\theta} + \frac{n-1}{i}\omega_1 - \omega_2\right)(t_1 - \tau) - \left(\frac{1}{\theta} - \frac{\omega_1}{i}\right)\sum_{j=2}^i (t_j - \tau)} dt_i \ldots dt_1 \\
&= \frac{n!\, e^{-n\frac{\tau-\mu}{\theta} + \tau\omega_2}}{(n-i)!\theta^i P[N=i]} \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{i-1}{j} (i-1)! \\
&\quad \times \frac{e^{\left(\frac{j+1}{\theta} + \frac{n-j-1}{i}\omega_1 - \omega_2\right)(\tau-\mu)} - 1}{\left(\frac{j+1}{\theta} + \frac{n-j-1}{i}\omega_1 - \omega_2\right)\left(\frac{1}{\theta} - \frac{\omega_1}{i}\right)^{i-1}}.
\end{aligned}
$$

Last equality is due to (2.13). Hence, for $i = r,\ r+1,\ \ldots,\ n$,

$$
\begin{aligned}
&E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}} \mid N = i] \times P[N=i] \\
&= e^{-n\frac{\tau-\mu}{\theta} + \tau\omega_2} \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{n}{i} \binom{i}{j+1} \frac{e^{\left(\frac{j+1}{\theta} + \frac{n-j-1}{i}\omega_1 - \omega_2\right)(\tau-\mu)} - 1}{\left(1 - \frac{\theta}{i}\omega_1\right)^{i-1}\left(1 + \frac{(n-j-1)\theta}{i(j+1)}\omega_1 - \frac{\theta}{j+1}\omega_2\right)}.
\end{aligned}
$$

Hence, using the above expression, (2.15) and (2.19) in (2.14), we have for $\omega_1 < \frac{r}{\theta}$ and $\omega_2 < \frac{n}{\theta}$

$$
\begin{aligned}
&E[e^{\omega_1 \widehat{\theta} + \omega_2 \widehat{\mu}}] \\
&= \frac{q^n e^{\tau\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_1}\right)^{\alpha_1}\left(1 - \frac{\omega_2}{\nu_{n-1}}\right)} + \frac{e^{\mu\omega_2} - q^n e^{\tau\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1 - \frac{\omega_2}{\nu_{n-1}}\right)} \\
&+ \sum_{i=1}^{n-1}\sum_{j=0}^{i-1} c_{ij} \frac{e^{\mu_{ij}\omega_1 + \mu\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_i}\right)^{\alpha_i}\left(1 + \frac{\omega_1}{\lambda_{ij}} - \frac{\omega_2}{\nu_j}\right)} - \sum_{i=1}^{n-1}\sum_{j=0}^{i-1} d_{ij} \frac{e^{\tau\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_i}\right)^{\alpha_i}\left(1 + \frac{\omega_1}{\lambda_{ij}} - \frac{\omega_2}{\nu_j}\right)} \\
&+ \sum_{j=0}^{n-2} c_{nj} \frac{e^{\mu_{nj}\omega_1 + \mu\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1 + \frac{\omega_1}{\lambda_{nj}} - \frac{\omega_2}{\nu_j}\right)} - \sum_{j=0}^{n-2} d_{nj} \frac{e^{\tau\omega_2}}{\left(1 - \frac{\omega_1}{\lambda_n}\right)^{\alpha_n}\left(1 + \frac{\omega_1}{\lambda_{nj}} - \frac{\omega_2}{\nu_j}\right)},
\end{aligned}
$$

where q, μ_{ij}, α_i, λ_i, λ_{ij}, c_{ij} and d_{ij} are as given in Theorem 2.3.1. Now using Lemmas 2.A.3, 2.A.4, one will get (2.9). ∎

Chapter 3

Bayesian Analysis of Different Hybrid and Progressive Life Tests[1]

3.1 Introduction

We have already discussed different censoring schemes in Section 1.2. Extensive work has been done on different censoring schemes by several authors (see Section 1.5). However, most of the analysis has been performed under the frequentist context and very little attention is paid to Bayesian analysis. Moreover, it is worth mentioning that the frequentist analysis of different HCSs and PCSs is not very easy even when lifetime distribution is assumed to be an exponential distribution. Though finding the MLE is not difficult, construction of associated CI involves a numerical computation. It seems that Bayesian approach is a natural choice in this case. Draper and Guttman [62] considered Bayesian analysis of hybrid life tests with one-parameter exponential failure times. The main aim of this chapter is to consider the Bayesian inference of the unknown parameters of a two-parameter exponential distribution when the data are obtained from different HCSs and PCSs. It is observed that the Bayesian analysis can be performed quite conveniently, and it can be extended quite naturally for many other censoring cases, which may not be that immediate in the

[1]A part of this work has been accepted for publication in *Communications in Statistics - Simulation and Computation.*

frequentist context.

Rest of the chapter is organized as follows. In Section 3.2, we briefly mention the model under consideration and prior information of the unknown parameters. In Section 3.3, we provide the posterior analysis and the Bayes estimators in details for HCS-I. Monte Carlo simulation results are presented in Section 3.4. In Section 3.5 we provide the analysis of a Type-I hybrid censored data set for illustrative purpose. In Section 3.6 we have indicated how the proposed method can be implemented for other censoring schemes also, and finally we conclude the chapter in Section 3.7.

3.2 Model Assumption and Prior Information

It is assumed that the failure times of the experimental units are independent and identically distributed two-parameter exponential random variables with the following PDF

$$f(t; \lambda, \mu) = \lambda e^{-\lambda(t-\mu)}; \quad t > \mu, \quad -\infty < \mu < \infty, \quad \lambda > 0.$$

We make the following prior assumptions on λ and μ. Note that for known μ, λ has a conjugate gamma prior and this prior was used by Draper and Guttman [62]. It is assumed that λ has a gamma distribution with the shape and scale parameters $a > 0$ and $b > 0$, respectively, *i.e.*, it has the following PDF

$$\pi_1(\lambda) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda}; \quad \lambda > 0.$$

It is further assumed that μ has a uniform prior over (M_1, M_2), where $M_1 < M_2$, *i.e.*, it has the following PDF

$$\pi_2(\mu) = \frac{1}{M_2 - M_1}; \quad M_1 < \mu < M_2.$$

It is also assumed that μ and λ are independently distributed. Note that if we take $a = b = 0$, it becomes a non-informative prior, which is an improper prior also on

λ. If we assume $M_1 \to -\infty$ and $M_2 \to \infty$, we have a non-informative and improper prior on μ over the whole real line.

3.3 Posterior Analysis under HCS-I

Let $r(\leq n)$ be a pre-chosen positive integer, and τ be a pre-determined time. Recall that under HCS-I the test is terminated when r-th item fails or time τ is reached on the test, whichever is earlier, *i.e.*, the termination time of the experiment is $\tau^* = \min\{t_{r:n}, \tau\}$. The available data will be one of the forms:

(a) $t_{1:n} < t_{2:n} < \ldots < t_{r:n}$ if $\tau^* = t_{r:n}$,

(b) $t_{1:n} < t_{2:n} < \ldots < t_{N:n}$ if $\tau^* = \tau$,

(c) there is no failure before the time τ,

where $N \in \{1, \ldots, r-1\}$ is the number of failures before the time τ. Note that the probability of getting no data depends on the chosen value of τ. During the planing stage one should not choose a value of τ such that the probability of getting no data is significant. Therefore we ignore case (c). Based on the observations from a HCS-I, the likelihood function can be written as

$$l(\text{Data} \,|\, \lambda, \mu) \propto \lambda^{d^*} e^{-\lambda\left(\sum_{i=1}^{d^*}(t_{i:n}-\mu)+(n-d^*)(U-\mu)\right)}.$$

Here for case (a), $d^* = r$, and $U = t_{r:n}$, and for case (b), $d^* = N$, $0 < N \leq r-1$, and $U = \tau$. Therefore, based on the priors $\pi_1(\cdot)$ and $\pi_2(\cdot)$ as mentioned above the joint posterior density function of λ and μ becomes

$$l(\mu, \lambda \,|\, \text{Data}) \propto \lambda^{a+d^*-1} e^{-\lambda(b+\sum_{i=1}^{d^*}(t_{i:n}-\mu)+(n-d^*)(U-\mu))}; \qquad \lambda > 0, \quad M_1 < \mu < M_3, \tag{3.1}$$

where $M_3 = \min\{M_2, t_{1:n}\}$. Let $D = \{(\mu, \lambda) : \lambda > 0, M_1 < \mu < M_3\}$. Note that $l(\mu, \lambda \,|\, \text{Data}) > 0$ for $(\mu, \theta) \in D$. Let $A_0 = \frac{1}{n}\left(b + \sum_{i=1}^{d^*} t_{i:n} + (n-d^*)U\right)$. Then for

$i = 1, 2$

$$A_0 - M_i = \frac{1}{n}\left\{b + \sum_{j=1}^{d^*}(t_{j:n} - M_i) + (n - d^*)(U - M_i)\right\} > 0,$$

as $t_{j:n} - M_i \geq 0$ for $i = 1, 2, \; j = 1, 2, \ldots, d^*$, $U - M_i \geq 0$ for $i = 1, 2$, and $b > 0$. Now

$$\int_0^\infty \int_{M_1}^{M_3} l(\mu, \lambda \,|\, \text{Data}) d\mu d\lambda$$

$$= (M_3 - M_1) \int_0^\infty \lambda^{a+d^*-1} \, e^{-n(A_0 - M_3)\lambda} \left\{\frac{1 - e^{-n(M_3 - M_1)\lambda}}{n(M_3 - M_1)\lambda}\right\} d\lambda,$$

which imply $l(\mu, \lambda \,|\, \text{Data})$ is integrable if $a + d^* > 0$ as

$$\lim_{\lambda \downarrow 0} \frac{1 - e^{-n(M_3 - M_1)\lambda}}{n(M_3 - M_1)\lambda} = 1.$$

Now consider the reverse integration

$$\int_{M_1}^{M_3} \int_0^\infty l(\mu, \lambda, \,|\, \text{Data}) d\lambda d\mu$$

$$= \int_{M_1}^{M_3} \frac{\Gamma(a + d^*)}{\{n(A_0 - \mu)\}^{a+d^*}} \quad \text{if } a + d^* > 0.$$

As $M_1 < M_3 < A_0$, the last integral exists and hence, $\displaystyle\int_{M_1}^{M_3} \int_0^\infty l(\mu, \lambda) d\lambda d\mu$ exists if $a + d^* > 0$. Using Fubini's Theorem, one can conclude that $l(\mu, \lambda \,|\, \text{Data})$ is integrable over the region $D = \{(\mu, \lambda) : \lambda > 0, M_1 < \mu < M_3\}$ if $a + d^* > 0$.

If we want to compute the Bayes estimate of some function of μ and λ, say $g(\mu, \lambda)$, with respect to the squared error loss function, it will be posterior expectation of $g(\mu, \lambda)$, *i.e.*,

$$\widehat{g}_B(\mu, \lambda) = \int_0^\infty \int_{M_1}^{M_3} g(\mu, \lambda) l(\mu, \lambda \,|\, \text{Data}) d\mu d\lambda, \tag{3.2}$$

provided it exists and is finite. Unfortunately (3.2) cannot be obtained in explicit form in most of the cases. Even when the integration (3.2) can be performed explicitly, it may not be possible to construct the corresponding CRI. For example, let us consider the p-th percentile point, say η_p, of the two-parameter exponential distribution, *i.e.*,

$$g(\mu, \lambda) = \eta_p = \mu - \frac{1}{\lambda} \ln(1-p).$$

The Bayes estimate of η_p with respect to the squared error loss function exists when $a + d^* - 1 > 0$ and $d^* > 0$, and it is

$$\widehat{\eta}_p = \begin{cases} A_0 - \frac{A_0 - M_1}{a + d^* - 2} \times \frac{\left(\frac{A_0 - M_1}{A_0 - M_3}\right)^{a+d^*-2} - 1}{\left(\frac{A_0 - M_1}{A_0 - M_3}\right)^{a+d^*-1} - 1} \times \{a + d^* - 1 + n \log(1-p)\} & \\ & \text{if} \quad a + d^* \neq 2 \\ A_0 - (A_0 - M_1)\left(\frac{A_0 - M_1}{A_0 - M_3} - 1\right)^{-1} (n \log(1-p) + 1) \log\left(\frac{A_0 - M_1}{A_0 - M_3}\right) & \\ & \text{if} \quad a + d^* = 2. \end{cases}$$

However, the posterior distribution of η_p cannot be obtained explicitly, and hence, finding the credible interval analytically is not a trivial issue. We propose to use the Monte Carlo sampling to construct the associated symmetric CRI of η_p.

Note that (3.1) can be written as

$$l(\mu, \lambda \,|\, \text{Data}) = l(\lambda \,|\, \mu, \text{Data}) \times l(\mu \,|\, \text{Data}), \tag{3.3}$$

where

$$\lambda \,|\, \{\mu, \text{Data}\} \sim \text{Gamma}\left(a + d^*,\ n(A_0 - \mu)\right), \tag{3.4}$$

$$l(\mu \,|\, \text{Data}) = \frac{c(a + d^* - 1)}{(A_0 - \mu)^{a+d^*}}; \quad M_1 < \mu < M_3,$$

$$c = \left\{ \frac{1}{(A_0 - M_3)^{a+d^*-1}} - \frac{1}{(A_0 - M_1)^{a+d^*-1}} \right\}^{-1}.$$

Moreover, the posterior distribution of $\mu \,|\, \text{Data}$ has a compact and invertible CDF as

$$F(x) = c \left\{ \frac{1}{(A_0 - x)^{a+d^*-1}} - \frac{1}{(A_0 - M_1)^{a+d^*-1}} \right\} \quad \text{for } M_1 \leq x < M_3 \tag{3.5}$$

and hence, random numbers can be generated quite easily from $l(\mu \,|\, \text{Data})$. Now we suggest to use the following procedure to compute the Bayes estimate of $g(\mu, \lambda)$, and also to construct the associated CRI.

Step 1. Generate μ_1 from (3.5).

Step 2. Generate λ_1 from $l(\lambda \mid \mu, \text{Data})$ as given in (3.4).

Step 3. Continue the process M times, and obtain $\{(\mu_1, \lambda_1), \cdots, (\mu_M, \lambda_M)\}$.

Step 4. Obtain $\{g(\mu_1, \lambda_1), \cdots, g(\mu_M, \lambda_M)\}$.

Step 5. Compute the Bayes estimate of $g(\mu, \lambda)$ as

$$\widehat{g}_{BE}(\mu, \lambda) = \frac{1}{M}\sum_{i=1}^{M} g(\mu_i, \lambda_i).$$

Step 6. To construct a 100(1-γ)% symmetric CRI of $g(\mu, \lambda)$, first order $g(\mu_i,\, \lambda_i)$'s, say $g_1 < g_2 < \cdots < g_M$, and obtain the symmetric CRI as $(g_{[M\gamma/2]},\, g_{[M(1-\gamma/2)]})$. Here $[x]$ denotes the largest integer less than or equal to x. HPD CRI is given by $(g_{j^*},\, g_{[M(1-\gamma)+j^*]})$, where j^* is such that

$$g_{[M(1-\gamma)+j^*]} - g_{j^*} \leq g_{[M(1-\gamma)+j]} - g_j \quad \text{for } j = 1,\, 2,\, \ldots,\, [M\gamma].$$

Similar methodology can be applied for other censoring schemes also, and we will briefly mention different cases in Section 3.6 for completeness purposes.

Next we shall examine BE and associate CRI when prior on μ is assumed to be a uniform non-informative over the whole real line $\mathbb{R}$, *i.e.*, $\pi_3(\mu) = d\mu$. Clearly, it is a non-proper prior. Other assumptions are as before. Therefore, based on the priors $\pi_1(\cdot)$ and $\pi_3(\cdot)$ the joint posterior density function of μ and λ becomes

$$\widetilde{l}(\mu, \lambda \mid \text{Data}) \propto \lambda^{d^*+a-1} e^{-\lambda(b+\sum_{i=1}^{d^*}(t_{i:n}-\mu)+(n-d^*)(U-\mu))}; \qquad \lambda > 0, \quad \mu < t_{1:n}. \qquad (3.6)$$

Note that (3.6) is a proper joint density function if $d^* + a - 1 > 0$. Like the previous section BE of $g(\alpha, \lambda)$ with respect to the squared error loss function is given by

$$\widehat{g}_B(\mu, \lambda) = \int_0^{\infty}\int_{-\infty}^{t_{1:n}} g(\mu, \lambda)\widetilde{l}(\mu, \lambda \mid \text{Data}) d\mu d\lambda, \qquad (3.7)$$

provided it exists and finite. Here also (3.7) cannot be obtained in explicit form in most of the cases. In this case the BE of η_p with respect to the squared error loss function exists when $a + d^* - 2 > 0$ and $d^* > 0$, and it is given by

$$\widehat{\eta}_p = \frac{(a + d^* - 1)t_{1:n} - A_0}{a + d^* - 2} - \frac{n(A_0 - t_{1:n})}{a + d^* - 2} \ln{(1 - p)}.$$

However, the posterior distribution of $\widehat{\eta}_p$ cannot be obtained explicitly in this case also, and hence, finding the CRI analytically is not a trivial issue. We propose to use the Monte Carlo random sampling technique to construct the associated symmetric CRI of $g(\mu,\,\lambda)$. Note that (3.6) can be written in the form of (3.3), where $l(\lambda\,|\,\mu,\,\text{Data})$ is given in (3.4) and

$$\begin{aligned} l(\mu\,|\,\text{Data}) &= \frac{\widetilde{c}}{(A_0-\mu)^{a+d^*}}\,; \quad \mu < t_{1:n}, \\ \widetilde{c} &= (a+d^*-1)\times(A_0-x_{1:n})^{a+d^*-1}\,. \end{aligned}$$

Moreover, the posterior distribution of $\mu\,|\,\text{Data}$ has a compact cumulative distribution function as

$$F(x) = \frac{(A_0-t_{1:n})^{a+d^*-1}}{(A_0-x)^{a+d^*-1}} \quad \text{for } \mu < t_{1:n}. \tag{3.8}$$

Therefore generation from $\mu\,|\,\text{Data}$ is very straight forward. Now we suggest to use the following procedure to compute the Bayes estimate of $g(\mu,\lambda)$, and also to construct the associated CRI.

Step 1. Generate μ_1 from (3.8).

Step 2. Generate λ_1 from $l(\lambda\,|\,\mu,\text{Data})$ as given in (3.4).

Step 3. Continue the process M times to obtain $\{(\mu_1,\lambda_1),\cdots,(\mu_M,\lambda_M)\}$, and also obtain $\{g(\mu_1,\lambda_1),\,\cdots,\,g(\mu_M,\lambda_M)\}$.

Step 4. Compute the Bayes estimate of $g(\mu,\lambda)$ as

$$\widehat{g}_{BE}(\mu,\lambda) = \frac{1}{M}\sum_{i=1}^{M} g(\mu_i,\lambda_i).$$

Step 5. To construct a 100(1-γ)% symmetric CRI of $g(\mu,\lambda)$, first order $g(\mu_i,\lambda_i)$ for $i=1,\,2,\,\ldots,\,M$, say $g_1<g_2<\cdots<g_M$, and obtain the symmetric CRI as $(g_{[M\gamma/2]},\,g_{[M(1-\gamma/2)]})$. Here $[x]$ denotes the largest integer less than or equal to x. HPD CRI is given by $(g_{j^*},\,g_{[M(1-\gamma)+j^*]})$, where j^* is such that

$$g_{[M(1-\gamma)+j^*]} - g_{j^*} \le g_{[M(1-\gamma)+j]} - g_j \quad \text{for } j=1,\,2,\,\ldots,\,[M\gamma].$$

3.4 Simulation Study

In this section we present some simulation results to examine how the proposed Bayes estimate and the associate credible interval behave for different sample sizes in case of HCS-I. We consider three parametric functions, *viz.* μ, λ, and $\eta_{0.90}$, and through out we assume $a = 2$, $b = 0.1$, $M_1 = -100$, $M_2 = 100$, and $n = 15$. In all these cases we have considered $\mu = 0$ and $\lambda = 10$. Note that Bayes estimate of μ and λ exist if $a + d^* > 0$ and they are given by

$$\widehat{\mu} = \begin{cases} A_0 + (M_3 - M_1)\left\{\ln\left(\frac{A_0 - M_3}{A_0 - M_1}\right)\right\}^{-1} & \text{if } a + d^* = 1 \\ A_0 + \frac{1}{(A_0 - M_3)} \ln\left(\frac{A_0 - M_3}{A_0 - M_1}\right)\left\{1 - \frac{A_0 - M_3}{A_0 - M_1}\right\} & \text{if } a + d^* = 2 \\ A_0 - \frac{a + d^* - 1}{a + d^* - 2}(A_0 - M_3) \times \frac{1 - \left(\frac{A_0 - M_3}{A_0 - M_1}\right)^{a + d^* - 2}}{1 - \left(\frac{A_0 - M_3}{A_0 - M_1}\right)^{a + d^* - 1}} & \text{otherwise} \end{cases}$$

and

$$\widehat{\lambda} = \frac{a + d^* - 1}{n(A_0 - M_3)} \times \frac{1 - \left(\frac{A_0 - M_3}{A_0 - M_1}\right)^{a + d^*}}{1 - \left(\frac{A_0 - M_3}{A_0 - M_1}\right)^{a + d^* - 1}}.$$

We computed the Bayes estimate both theoretically and by Monte Carlo sampling. We computed the 90%, 95% and 99% symmetric CRIs using Monte Carlo sampling as suggested in the previous section. We report the AE and MSE of the Bayes estimates, and the CPs, the ALs of symmetric CRIs based on 5000 replications and $M = 5000$ in each case. The results are reported in Tables 3.1-3.3.

It is observed that in each case as τ increases the biases and the MSEs decrease, it verifies the consistency properties of the estimates. In all the cases the coverage percentages are also quite close to the nominal levels even for very small sizes. For fixed sample size n and effective sample size, the coverage percentages decrease as τ increases. For fixed n, as r increases the biases, MSEs and the length of the credible intervals decrease. Moreover, in all the cases the theoretical and simulated Bayes estimates match very well.

Table 3.1: Average estimates and the corresponding MSEs of μ and average lengths of the three different credible intervals and the associated coverage percentages

r	τ	Sim.		Theo.		90%		95%		99%	
		AE	MSE	AE	MSE	CP	AL	CP	AL	CP	AL
5	0.050	-0.00065	0.0001	-0.00041	0.0001	89.72	0.024	95.00	0.032	99.00	0.056
	0.075	-0.00015	0.0001	0.00007	0.0001	89.30	0.022	94.46	0.029	99.02	0.049
	0.100	-0.00008	0.0001	0.00016	0.0001	88.70	0.021	93.84	0.028	98.48	0.047
	0.125	-0.00009	0.0001	0.00018	0.0001	88.56	0.021	94.40	0.028	98.78	0.047
	0.150	-0.00009	0.0001	0.00018	0.0001	88.56	0.021	94.40	0.028	98.78	0.047
10	0.050	-0.00113	0.0001	-0.00083	0.0001	90.12	0.025	95.12	0.033	99.02	0.057
	0.075	-0.00061	0.0001	-0.00041	0.0001	90.24	0.023	95.28	0.030	99.08	0.049
	0.100	-0.00039	0.0000	-0.00015	0.0001	89.52	0.022	94.88	0.028	98.94	0.044
	0.125	-0.00017	0.0001	0.00001	0.0001	88.88	0.021	93.70	0.027	98.64	0.043
	0.150	-0.00006	0.0000	0.00010	0.0001	89.48	0.021	94.28	0.027	98.96	0.042
15	0.050	-0.00112	0.0001	-0.00084	0.0001	90.28	0.024	95.08	0.032	99.08	0.055
	0.075	-0.00052	0.0001	-0.00045	0.0001	90.06	0.023	94.76	0.030	98.90	0.048
	0.100	-0.00049	0.0000	-0.00026	0.0001	90.46	0.022	95.32	0.028	98.76	0.045
	0.125	-0.00024	0.0000	-0.00015	0.0001	89.48	0.022	94.52	0.028	98.98	0.044
	0.150	-0.00026	0.0000	-0.00009	0.0001	89.76	0.021	94.92	0.027	98.96	0.042

Table 3.2: Average estimates and the corresponding MSEs of λ and average lengths of the three different credible intervals and the associated coverage percentages

r	τ	Sim.		Theo.		90%		95%		99%	
		AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL
5	0.050	13.78	49.0096	13.93	51.4781	90.48	18.519	96.12	22.185	99.72	29.527
	0.075	13.86	48.7823	14.07	50.9108	90.02	18.285	96.24	21.897	99.70	29.104
	0.100	14.09	51.6411	14.09	50.7609	90.08	18.553	95.76	22.217	99.62	29.533
	0.125	14.11	52.0124	14.09	50.7301	90.18	18.579	96.06	22.249	99.52	29.578
	0.150	14.11	52.0124	14.09	50.7285	90.18	18.579	96.06	22.249	99.52	29.578
10	0.050	11.95	20.6322	12.15	22.7665	92.74	14.550	96.18	17.415	99.22	23.129
	0.075	11.69	18.0393	11.91	20.7534	90.38	12.648	95.88	15.113	99.26	20.007
	0.100	11.97	19.9217	11.98	20.8835	89.60	12.163	94.88	14.530	99.28	19.212
	0.125	12.01	19.9753	12.07	20.8048	89.02	11.928	95.02	14.248	99.10	18.823
	0.150	11.97	18.6423	12.12	20.6603	90.22	11.791	95.02	14.078	99.00	18.596
15	0.050	12.09	21.3983	12.10	21.4073	92.04	14.628	96.02	17.506	99.28	23.253
	0.075	11.64	16.0866	11.71	16.5803	91.10	12.481	95.92	14.916	99.22	19.739
	0.100	11.47	13.6284	11.54	14.4483	90.82	11.373	95.62	13.587	99.26	17.948
	0.125	11.36	12.4393	11.46	13.2163	90.96	10.700	95.40	12.777	99.04	16.879
	0.150	11.41	12.4330	11.43	12.6241	90.26	10.342	94.92	12.346	98.94	16.292

3.5 Data Analysis

For illustrative purposes, we present the analysis of HCS-I data set. The data set has been obtained from Bain and Englehardt [8]. In this case 20 items were put on

Table 3.3: AE and the corresponding MSE of $\eta_{0.90}$ and AL of the three different CRIs and the associated CP

r	τ	Sim.		Theo.		90%		95%		99%	
		AE	MSE	AE	MSE	CP	AL	CP	AL	CP	AL
5	0.050	0.25	0.0226	0.25	0.0227	90.46	0.394	96.16	0.517	99.72	0.865
	0.075	0.24	0.0099	0.24	0.0100	89.76	0.334	96.22	0.428	99.70	0.671
	0.100	0.23	0.0089	0.23	0.0089	89.88	0.324	95.60	0.414	99.62	0.646
	0.125	0.23	0.0083	0.23	0.0083	90.14	0.322	95.92	0.411	99.50	0.640
	0.150	0.23	0.0083	0.23	0.0083	90.14	0.322	95.92	0.411	99.50	0.640
10	0.050	0.27	0.0186	0.27	0.0187	92.60	0.389	96.30	0.506	99.12	0.827
	0.075	0.25	0.0107	0.25	0.0107	90.16	0.308	95.88	0.389	99.16	0.588
	0.100	0.24	0.0071	0.24	0.0071	89.74	0.262	94.74	0.326	99.24	0.478
	0.125	0.24	0.0064	0.24	0.0064	89.22	0.246	95.12	0.306	99.04	0.442
	0.150	0.23	0.0055	0.23	0.0055	90.20	0.238	94.86	0.295	98.92	0.424
15	0.050	0.26	0.0161	0.26	0.0161	91.96	0.377	96.06	0.488	99.24	0.791
	0.075	0.25	0.0095	0.25	0.0095	91.08	0.303	95.98	0.382	99.24	0.575
	0.100	0.24	0.0067	0.24	0.0068	90.96	0.265	95.66	0.330	99.28	0.482
	0.125	0.24	0.0064	0.24	0.0064	90.94	0.246	95.50	0.304	98.92	0.439
	0.150	0.24	0.0051	0.24	0.0051	90.08	0.228	94.86	0.281	98.88	0.401

a life test and they were observed for 150 hours. During that period 13 items failed with the following lifetime, measured in hours,: 3, 19, 23, 26, 37, 38, 41, 45, 58, 84, 90, 109, and 138. In this case $n = 20$, $r = 13$, and $\tau = 150$.

For this data set we obtain the Bayes estimates of $\eta_{0.90}$ and the associated symmetric CRI with $a = 5$, $b = 0.1$, $M_1 = -100$, and $M_2 = 100$. The results are as follows: $\widehat{\eta}_{0.90}$(Theoretical) $= 374.98$ and $\widehat{\eta}_{0.90}$(Simulation) $= 376.22$. The associated 90%, 95%, and 99% symmetric CRI are (245.58, 561.08), (229.70, 615.92), and (199.24, 751.54) respectively. If $a = 2$, $b = 0.1$, $M_1 = 0$, and $M_2 = 3$, the Bayes estimate of $\eta_{0.90}$ is 363.03 (theoretical) and 364.00 (simulated). The associated 90%, 95%, and 99% symmetric credible intervals are (228.19, 559.13), (213.48, 608.33), and (186.72, 770.18) respectively.

Now we will provide the empirical Bayes estimator of $\eta_{0.90}$. Note that in empirical Bayes analysis a popular choice of the hyper-parameters are argument maximum of the integrated posterior density function. In this case for $a + d^* > 1$, the integrated posterior density function, say $I(a,\, b,\, M_1,\, M_2)$, can be written as

$$\begin{aligned} I(a,\, b,\, M_1,\, M_2) &= \iint_D l(\mu,\, \lambda \,|\, \text{Data}) d\lambda d\mu \\ &= \frac{b^a\, \Gamma(a+d^*-1)}{n\,(M_1-M_2)\,\Gamma(a)} \times \left[\frac{1}{(b+A_1)^{a+d^*-1}} - \frac{1}{(b+A_2)^{a+d^*-1}} \right], \end{aligned}$$

where $D = \{(\mu,\, \lambda) : M_1 < \mu < M_3,\, \lambda > 0\},\ A_1 = \sum_{i=1}^{d^*} t_{i:n} + (n-d^*)U - nM_3$, and $A_2 = \sum_{i=1}^{d^*} t_{i:n} + (n-d^*)U - nM_1$. Here we assume M_1 and M_2 are known and want to maximize $I(a,\, b,\, M_1,\, M_2)$ with respect to a and b only. When M_1 and M_2 are known, we denote $I(a,\, b,\, M_1,\, M_2)$ by $I(a,\, b)$ for simplicity. For fixed a, the value of b, say $b^*(a)$, which maximizes the integrated posterior density function, is a positive solution of the equation

$$h(x) = 0,$$

where

$$\begin{aligned} h(x) &= a(x+A_1)(x+A_2)^{a+d^*} - a(x+A_1)^{a+d^*}(x+A_2) \\ &\quad + (a+d^*-1)x(x+A_1)^{a+d^*} - (a+d^*-1)x(x+A_2)^{a+d^*}. \end{aligned}$$

Analytically we could not prove that $I(a,\, b)$ does not have a maximum for finite (a, b). However, the contour plot of $\log\{I(a,\, b)\}$ (see Figure 3.1), suggests that $I(a,\, b)$ does not possess a maximum.

Next empirical Bayes estimator of $\eta_{0.90}$ is considered when prior $\pi_3(\cdot)$ is assumed on the parameter μ. In this case the integrated posterior density function exists if $a + d^* - 1 > 0$, and is given by

$$\widetilde{I}(a,\, b) = \frac{b^a\, \Gamma(a+d^*-1)}{n\, \Gamma(a)} \times \frac{1}{(b+A_1)^{a+d^*-1}}. \tag{3.9}$$

In this case for fixed a, the value of b, say $b^*(a)$, which maximizes (3.9), is given by

$$b^*(a) = \frac{Aa}{d^*-1}, \tag{3.10}$$

when $d^* > 1$. It can be shown, see in the Appendix, that $\widetilde{I}(a, b^*(a))$ is an increasing function of a. Contour plot of $\log\{\widetilde{I}(a,\,b)\}$ is reported in the Figure 3.1 along with the contour plot of $\log\{I(a,\,b)\}$ with $M_1 = -100$ and $M_2 = 100$. These two contour plots are not distinguishable as we take quite large range for the prior distribution of μ.

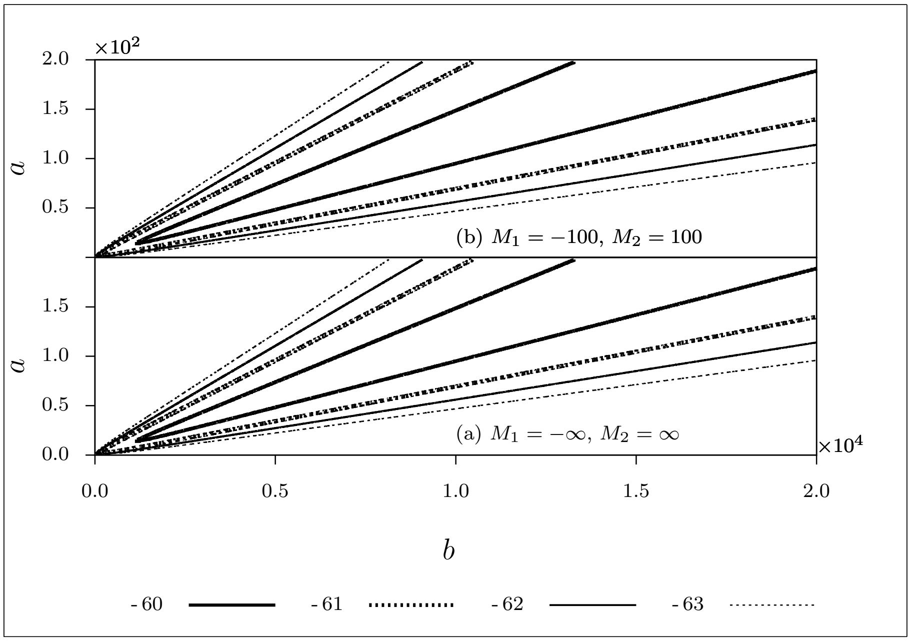


Figure 3.1: Contour plots of the logarithm of the integrated posterior density functions.

3.6 Posterior Analysis under Other CSs

The results corresponding to HCS-II, GHCS-I, and GHCS-II (see Section 1.2.2) can be obtained in a very similar way as the HCS-I. Now we will briefly discuss the Bayesian inference of the unknown parameters based on the observations obtained from different progressive censoring schemes.

Type-I Progressive Censoring Scheme

Based on the observations from a PCS-I (see Section 1.2.3), the likelihood function can be written as

$$l(\text{Data} \,|\, \lambda, \mu) \propto \lambda^k e^{-\lambda W(\mu)},$$

here

$$W(\mu) = \sum_{i=1}^{k}(t_{i:n} - \mu) + \sum_{j=1}^{k} R_j(\tau_j - \mu) = \sum_{i=1}^{k} t_{i:n} + \sum_{j=1}^{k} R_j\tau_j - n\mu.$$

The posterior density function of λ and μ can be written as

$$l(\mu, \lambda \,|\, \text{Data}) = l(\lambda \,|\, \mu, \text{Data}) \times l(\mu \,|\, \text{Data}); \quad \lambda > 0,\ M_1 < \mu < M_3,$$

here

$$\lambda \,|\, \{\mu,\ \text{Data}\} \sim \text{Gamma}\,(a + k, b + W(\mu))\,,$$

$$l(\mu \,|\, \text{Data}) = \frac{c_1\,(a + k - 1)}{(A_3 - \mu)^{a+m}}; \quad M_1 < \mu < M_3,$$

where

$$A_3 = \frac{1}{n} \times \left(b + \sum_{i=1}^{k} t_{i:n} + \sum_{j=1}^{k} R_j T_j\right),$$

and

$$c_1 = \left\{\frac{1}{(A_3 - M_3)^{a+m-1}} - \frac{1}{(A_3 - M_1)^{a+m-1}}\right\}^{-1}.$$

Therefore, in this case also the Bayes estimate and the associated credible interval can be constructed in a very similar way.

Type-II Progressive Censoring Scheme

Based on the data obtained from a PCS-II (see Section 1.2.3), the likelihood function in this case can be written as

$$l(\text{Data} \,|\, \lambda,\ \mu) \propto \lambda^k e^{-\lambda W(\mu)},$$

here

$$W(\mu) = \sum_{i=1}^{k}(t_{i:n} - \mu) + \sum_{i=1}^{k} R_j(t_{i:n} - \mu) = \sum_{i=1}^{k}(R_i + 1)t_{i:n} - n\mu.$$

The posterior density function of λ and μ can be written as

$$l(\mu, \lambda \,|\, \text{Data}) = l(\lambda \,|\, \mu, \text{Data}) \times l(\mu \,|\, \text{Data}); \quad \lambda > 0,\ M_1 < \mu < M_3,$$

where

$$\lambda \,|\, \{\mu, \text{Data}\} \sim \text{Gamma}\,(a+k, b+W(\mu))\,,$$

$$l(\mu \,|\, \text{Data}) = \frac{c_2\,(a+k-1)}{(A_4-\mu)^{a+k}}; \quad M_1 < \mu < M_3,$$

$$A_4 = \frac{1}{n} \times \left(b + \sum_{i=1}^{k}(R_i+1)t_{i:n}\right),$$

and

$$c_2 = \left\{\frac{1}{(A_4-M_3)^{a+k-1}} - \frac{1}{(A_4-M_1)^{a+k-1}}\right\}^{-1}.$$

Therefore, in this case also the Bayes estimate and the associated credible interval can be constructed in a very similar way.

Progressive Hybrid Type-II Censoring Scheme

Based on the observations from a PHCS-II (see Section 1.2.3), the likelihood function can be written as

$$l(\text{Data} \,|\, \lambda, \mu) \propto \lambda^{d^*} e^{-\lambda W(\mu)},$$

where for Case (a), $d^* = k$ and $W(\mu) = \sum_{i=1}^{k}(1+R_i)(t_{i:n}-\mu)$, and for Case (b), $d^* = N$, and $W(\mu) = \sum_{i=1}^{N}(1+R_i)(t_{i:n}-\mu) + (\tau-\mu)R_N^*$. The posterior density function of λ and μ can be written as

$$l(\mu,\ \lambda \,|\, \text{Data}) = l(\lambda \,|\, \mu,\ \text{Data}) \times l(\mu \,|\, \text{Data}); \quad \lambda > 0,\ M_1 < \mu < M_3,$$

where

$$\lambda \,|\, \{\mu, \text{Data}\} \sim \text{Gamma}\,(a+d^*,\ b+W(\mu))\,,$$

$$l(\mu \,|\, \text{Data}) = \frac{c_3\,(a+d^*-1)}{(A_5-\mu)^{a^*+d}}; \quad M_1 < \mu < M_3,$$

$$A_5 = \begin{cases} \frac{1}{n} \times \left(b + \sum_{i=1}^{k}(1+R_i)t_{i:n}\right) & \text{for} \quad \text{Case (a)} \\ \frac{1}{n} \times \left(b + \sum_{i=1}^{N}(1+R_i)t_{i:n} + R_N^* \tau\right) & \text{for} \quad \text{Case (b),} \end{cases}$$

and

$$c_3 = \left\{ \frac{1}{(A_5 - M_3)^{a+d^*-1}} - \frac{1}{(A_5 - M_1)^{a+d^*-1}} \right\}^{-1}.$$

Therefore, in this case also the Bayes estimate and the associated credible interval can be constructed in a very similar way.

3.7 Conclusion

In this chapter we have considered the Bayesian inference of the two-parameter exponential model when the data are hybrid or progressively censored. We have assumed a uniform prior on the location parameter and gamma prior on the scale parameter. The Bayes estimates may not be obtained explicitly in many cases, even when they exist. We have suggested to use the Monte Carlo sampling to compute simulation consistent Bayes estimators and also to construct the credible intervals. Monte Carlo simulation results suggest that the proposed Bayes estimators work quite well.

3.A Appendix

In this section we provide a formal proof that $J(a) = n\widetilde{I}(a, b^*(a))$, where $b^*(a) = \frac{Aa}{d^*-1}$, is an increasing function of a. Now we will show that $\log J(a)$ is an increasing function of a. Let us consider

$$\frac{d \ln J(a)}{da} = \sum_{i=0}^{d^*-2} \frac{1}{a+i} - \log\left(1 + \frac{d^*-1}{a}\right). \tag{3.11}$$

We will show that the right hand side of (3.11) is positive and we will show it by induction on d^*. Note that for $d^* = 2$, the right hand side of (3.11) is clearly positive. Now consider $d^* = 3$, and let

$$f(a) = \log(1 + \frac{2}{a}) - \log(1 + \frac{1}{a}) - \frac{1}{a+1}.$$

Using $x = \frac{1}{a}$, consider the function

$$g(x) = f\left(\frac{1}{a}\right) = \log(1 + 2x) - \log(1 + x) - \frac{x}{x+1},$$

therefore for $x \geq 0$,

$$g'(x) = \frac{2}{1+2x} - \frac{1}{1+x} - \frac{1}{(1+x)^2} = -\frac{x}{(1+2x)(1+x)^2} \leq 0.$$

This implies that $g(x)$ is a decreasing function of x for $x \geq 0$. Since $g(0) = 0$, $g(x) \leq 0$ for $x \geq 0$. Moreover, since $\log(1 + x) \leq x$, for $x \geq 0$, we have

$$\log(1 + 2x) \leq \log(1 + x) + \frac{x}{1+x} \leq x + \frac{x}{1+x}. \tag{3.12}$$

From (3.12) it immediately follows

$$\frac{1}{a} + \frac{1}{a+1} - \log(1 + \frac{2}{a}) \geq 0.$$

Hence, $\log J(a)$ is an increasing function of a for $d^* = 3$. Let it be true for $d^* = m$ and will prove that it is true for $d^* = m + 1$ also. Let

$$f_m(a) = \log(1 + \frac{m-1}{a}) - \log(1 + \frac{1}{a}) - \sum_{i=1}^{m-2} \frac{1}{a+i}$$

$$f_{m+1}(a) = \log(1 + \frac{m}{a}) - \log(1 + \frac{1}{a}) - \sum_{i=1}^{m-1} \frac{1}{a+i} = f_m(a) + h_m(a),$$

where

$$h_m(a) = \log\left(1 + \frac{m}{a}\right) - \log\left(1 + \frac{m-1}{a}\right) - \frac{1}{a+m-1}.$$

Using $x = \frac{1}{a}$ we consider the new function

$$g_m(x) = h_m\left(\frac{1}{a}\right) = \log(1 + mx) - \log(1 + (m-1)x) - \frac{x}{1+(m-1)x}.$$

Since for $x \geq 0$,

$$g'_m(x) = \frac{m}{1+mx} - \frac{m-1}{1+(m-1)x} - \frac{1}{\left(1+(m-1)x\right)^2} = \frac{-x}{(1+mx)(1+(m-1)x)} \leq 0,$$

$g_m(x)$ is a decreasing function of x. As $g_m(0) = 0$, $g_m(x) \leq 0$ for all $x \geq 0$, $h_m(a) \leq 0$ for $a \geq 0$. Since $f_m(a) \leq 0$ due to induction hypothesis, $f_{m+1}(a) \leq 0$. Therefore,

$$\log\left(1 + \frac{m}{a}\right) \leq \log\left(1 + \frac{1}{a}\right) + \sum_{i=1}^{m-1} \frac{1}{a+i} \leq \sum_{i=0}^{m-1} \frac{1}{a+i},$$

hence,

$$\sum_{i=0}^{m-1} \frac{1}{a+i} - \log\left(1 + \frac{m}{a}\right) \geq 0. \qquad \blacksquare$$

Chapter 4

Order Restricted Bayesian Inference for Exponential Step-stress Model

4.1 Introduction

In the last two chapters we have considered the frequentist and Bayesian analysis of a two parameter exponential model under different censoring schemes. In this chapter we consider a simple step-stress model, and it is assumed that the lifetimes are exponentially distributed with mean λ_1^{-1} and λ_2^{-1} at the stress level s_1 and s_2, respectively. The analysis has been performed based on CEM assumptions. Simple step-stress models under different censoring schemes are extensively studied based on the assumption that the lifetime of the experimental units follow exponential distributions (with different scale parameters) at different stress levels. In most of the cases CEM has been assumed. Interested readers are referred to the review article by Balakrishnan [10] in this respect. In all these cases the exact distributions of the unknown parameters are obtained, and they can be used to construct exact CIs. However, it is observed that the exact distribution and therefore the construction of associated CI is quite complicated in all these cases. It may be mentioned that although extensive amount of work has been done on step-stress models, not much attention has been paid to develop the inference imposing the restriction $\lambda_1 < \lambda_2$, which is a very natural choice for a simple SSLT. Balakrishnan et al. [13] considered the order restricted inference for step-stress models when the lifetimes are independently and exponentially distributed and the data are Type-I or Type-II censored. They have mainly adopted the frequentist approach, and the MLEs of the unknown parameters are obtained using isotonic regression. It is observed that obtaining the exact joint distribution of the MLEs is not very easy, hence, they derived the asymptotic distribution of the MLEs. Based on the asymptotic distribution, the asymptotic CIs of the unknown parameters can be constructed. It is not immediate that how this method can be extended for more general censoring schemes. It seems that Bayesian analysis is a natural choice in these cases. Though some work has been done on the Bayesian inference of the step-stress model, see for example Drop et al. [64], Lee and Pan [94], Leu and Shen [95] or Fan et al. [71], none of them dealt with the ordered restricted inference.

The main aim of this chapter is to consider the order restricted Bayesian inference of the unknown parameters of a simple step-stress model under different censoring schemes when the lifetimes of the experimental units are assumed to be exponentially distributed. We have assumed fairly flexible priors on the unknown parameters. It is observed that in all the cases the Bayes estimates of the unknown parameters cannot be obtained in explicit form. We propose to use importance sampling technique to compute Bayes estimate and also to construct associated CRI. Extensive Monte Carlo simulations are performed to see the effectiveness of the proposed method in case of CS-I, and the performances are quite satisfactory. The analysis of two data sets have been performed for illustrative purposes.

Rest of the chapter is organized as follows. Model assumptions and the prior information of the unknown parameters are considered in Section 4.2. In Section 4.3, maximum likelihood estimation of the unknown parameters imposing the order restriction on them is briefly discussed, when data are Type-I censored. In Section 4.4, we provide the posterior analysis and the Bayes estimators in details in case of CS-I. Monte Carlo simulation results and data analysis are presented in Section 4.5. In Section 4.6, we have indicated how the proposed method can be implemented for other censoring schemes. Finally, we conclude the chapter in Section 4.7.

4.2 Model Assumption and Prior Information

We consider a simple SSLT, where n identical units are placed on a life testing experiment at the initial stress level s_1. The stress level is increased to a higher level s_2 at a prefixed time τ_1. It is assumed that the lifetimes of the experimental units are independently and exponentially distributed random variables with different scale parameters at different stress levels. PDF and the CDF of the lifetime under stress level s_i for $i = 1,\, 2$, is given by

$$f(t; \lambda_i) = \lambda_i\, e^{-\lambda_i t} \qquad \text{for} \quad 0 < t < \infty \qquad \lambda_i > 0 \tag{4.1}$$

and

$$F(t; \lambda_i) = 1 - e^{-\lambda_i t} \qquad \text{for} \quad 0 < t < \infty \qquad \lambda_i > 0, \tag{4.2}$$

respectively. Let us assume that the stress level is changed from s_1 to s_2 at the time point τ_1 and τ_2 be the time of the termination of the experiment. It is further assumed that the failure time data comes from a CEM, hence, it has the following CDF;

$$G(t;\, \lambda_1,\, \lambda_2) = \begin{cases} F(t; \lambda_1) & \text{if} \quad 0 < t \le \tau_1 \\ F\left(t - \left(1 - \frac{\lambda_1}{\lambda_2}\right)\tau_1; \lambda_2\right) & \text{if} \quad \tau_1 < t < \infty. \end{cases} \tag{4.3}$$

The corresponding PDF is given by

$$g(t;\, \lambda_1,\, \lambda_2) = \begin{cases} \lambda_1 e^{-\lambda_1 t} & \text{if} \quad 0 < t \le \tau_1 \\ \lambda_2 e^{-\lambda_2 (t + \frac{\lambda_1}{\lambda_2}\tau_1 - \tau_1)} & \text{if} \quad \tau_1 < t < \infty. \end{cases} \tag{4.4}$$

For developing the Bayesian inference, we need to assume some priors on the unknown parameters. We want the prior assumption on λ_1 and λ_2, so that it maintains the order restriction, namely, $\lambda_1 < \lambda_2$. We take the following priors on λ_1 and λ_2. It is assumed that λ_2 has a Gamma(a, b) distribution with $a > 0$ and $b > 0$, *i.e.*, it has the following PDF

$$\pi_1(\lambda_2) = \frac{b^a}{\Gamma(a)} \lambda_2^{a-1} e^{-b\lambda_2} \quad \text{for} \quad \lambda_2 > 0. \tag{4.5}$$

Moreover, $\lambda_1 = \alpha\,\lambda_2$ and α has a beta distribution with parameters $c > 0$ and $d > 0$, *i.e.*, the PDF of α is given by

$$\pi_2(\alpha) = \frac{1}{B(c,d)} \alpha^{c-1}(1-\alpha)^{d-1} \quad \text{for} \quad 0 < \alpha < 1, \tag{4.6}$$

and the distribution of α is independent of λ_2. Therefore, the joint prior of (λ_1, λ_2) can be written as;

$$\pi(\lambda_1, \lambda_2) = \frac{b^a}{\Gamma(a)B(c,d)} \lambda_2^{a-c-d} e^{-b\lambda_2} \lambda_1^{c-1}(\lambda_2 - \lambda_1)^{d-1} \quad \text{for} \quad 0 < \lambda_1 < \lambda_2 < \infty. \tag{4.7}$$

As the joint prior on $(\lambda_1,\, \lambda_2)$ is little complicated, a gray-scale plot is provided in Figure 4.1 for different values of hyper-parameters. In the plot black color represents the maximum value of density function, whereas white color represents the minimum value, which is zero in all the plots. We have taken $b = 1.0$ only, as different values of b only effects the spread of the density function keeping the shape fix.

4.3 Maximum Likelihood Estimator under CS-I

In this section we present maximum likelihood estimation of the scale parameters under the restriction $\lambda_1 \leq \lambda_2$, when data are Type-I censored. Recall that the form of the ordered observed data under CS-I can have one of the following forms.

(a) $\tau_1 < t_{1:n} < \ldots < t_{n_2:n} < \tau_2$,

(b) $t_{1:n} < \ldots < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < \ldots < t_{n_1+n_2:n} < \tau_2$,

(c) $t_{1:n} < \ldots < t_{n_1:n} < \tau_1 < \tau_2$.

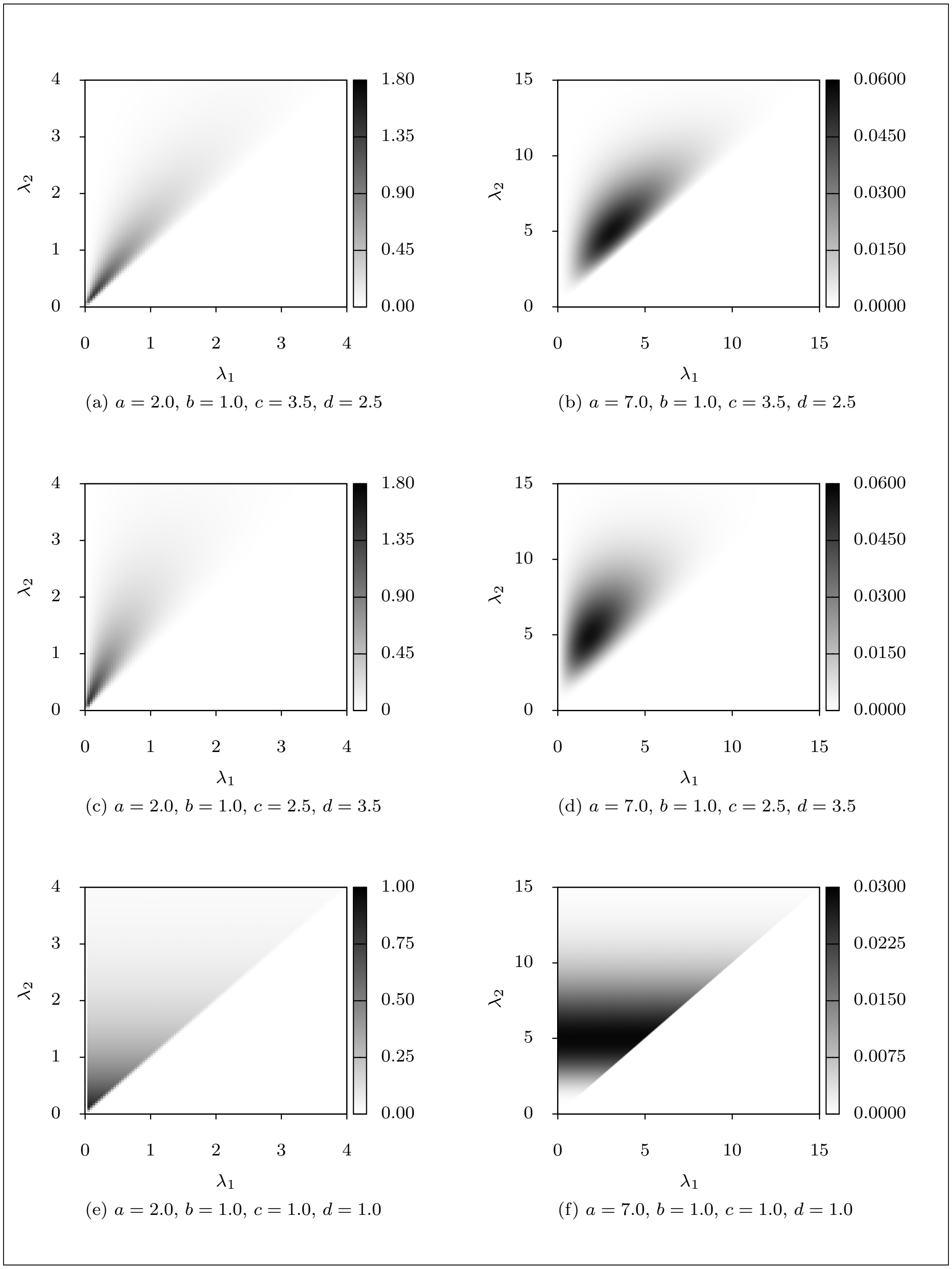


Figure 4.1: Plot of prior density for different values of hyper-parameters.

Let n_1^* and n_2^* be the number of failures at the stress level s_1 and s_2, respectively. Let τ^* be the termination time of the experiment. In case of CS-I, $\tau^* = \tau_2$. For Case (a): $n_1^* = 0$, $n_2^* = n_2 \leq n$, Case (b): $n_1^* = n_1 > 0$, $n_2^* = n_2 > 0$, Case (c): $n_1^* = n_1 > 0$, $n_2^* = 0$. In all the cases $n^* = n_1^* + n_2^*$. Based on the observations from a simple SSLT under CS-I, the likelihood can be written as

$$l_1(\lambda_1, \lambda_2 \,|\, \text{Data}) \;=\; \lambda_1^{n_1^*} \lambda_2^{n_2^*} e^{-\lambda_1 d_1 - \lambda_2 d_2}, \tag{4.8}$$

where $d_1 = \sum_{j=1}^{n_1^*} t_{j:n} + (n - n_1^*)\tau_1$, $d_2 = \sum_{j=n_1^*+1}^{n^*} (t_{j:n} - \tau_1) + (n - n^*)(\tau^* - \tau_1)$. Note that d_1 and d_2 are the total time elapsed by all the units at stress level s_1 and s_2, respectively. The unrestricted MLE of λ_1 and λ_2 is given by

$$\widehat{\lambda}_1^* = \frac{n_1^*}{d_1} \qquad \text{and} \qquad \widehat{\lambda}_2^* = \frac{n_2^*}{d_2}.$$

Clearly, if $\widehat{\lambda}_1^* \leq \widehat{\lambda}_2^*$, MLE of the scale parameters under the restriction $\lambda_1 \leq \lambda_2$ is given by

$$\widehat{\lambda}_1 = \widehat{\lambda}_1^* = \frac{n_1^*}{d_1} \qquad \text{and} \qquad \widehat{\lambda}_1 = \widehat{\lambda}_1^* = \frac{n_2^*}{d_2}.$$

As $l_1(\lambda_1, \lambda_2 \,|\, \text{Data})$ is unimodal function, if $\widehat{\lambda}_1^* > \widehat{\lambda}_2^*$, maximization of $l_1(\lambda_1, \lambda_2 \,|\, \text{Data})$ under the order restriction $\lambda_1 \leq \lambda_2$ is equivalent to maximization of $l_1(\lambda_1, \lambda_2 \,|\, \text{Data})$ under $\lambda_1 = \lambda_2$, and hence, in this case the MLEs of the scale parameters under the restriction $\lambda_1 \leq \lambda_2$ is given by

$$\widehat{\lambda}_1 = \widehat{\lambda}_2 = \frac{n_1^* + n_2^*}{d_1 + d_2}.$$

4.4 Posterior Analysis under CS-I

Based on the likelihood function in (4.8), priors $\pi_1(\cdot)$ and $\pi_2(\cdot)$ mentioned in Section 4.2, posterior density function of (α, λ_2) becomes

$$l_2(\alpha, \lambda_2 \,|\, \text{Data}) \;\propto \alpha^{n_1^*+c-1} (1-\alpha)^{d-1} \lambda_2^{n^*+a-1} e^{-\lambda_2 (d_1 \alpha + d_2 + b)} \quad \text{if } 0 < \alpha < 1,\ \lambda_2 > 0. \tag{4.9}$$

The right hand side of (4.9) is integrable if $n_1^*+c > 0$ and $n^*+a > 0$. Bayes estimate of some function of α and λ_2, say $g(\alpha,\, \lambda_2)$, with respect to the squared error loss function, is posterior expectation of $g(\alpha,\, \lambda_2)$, *i.e.*,

$$\widehat{g}(\alpha,\, \lambda_2) = \int_0^1 \int_0^\infty g(\alpha,\, \lambda_2) l_2(\alpha, \lambda_2 \,|\, \text{Data}) \mathrm{d}\lambda_2 \mathrm{d}\alpha. \tag{4.10}$$

Unfortunately, the close form of (4.10) cannot be obtained in most of the cases. One may use numerical techniques to compute (4.10). Alternatively, other approximation can be used to compute (4.10). However, CRI for a parametric function cannot be constructed by these numerical methods. So here we propose to use importance sampling to compute Bayes estimate as well as to construct CRI of a parametric function. Note that for $0 < \alpha < 1$ and $\lambda_2 > 0$, $l_2(\alpha,\, \lambda_2 \,|\, \text{Data})$ can be expressed as

$$l_2(\alpha,\, \lambda_2 \,|\, \text{Data}) = l_3(\alpha \,|\, \text{Data}) \times l_4(\lambda \,|\, \lambda_2,\, \text{Data}), \tag{4.11}$$

where

$$l_3(\alpha \,|\, \text{Data}) = c_1 \frac{\alpha^{n_1^*+c-1}(1-\alpha)^{d-1}}{(d_1\alpha + d_2 + b)^{a+n^*}}, \tag{4.12}$$

and

$$l_4(\lambda_2 \,|\, \alpha,\, \text{Data}) = \frac{\{d_1\alpha + d_2 + b\}^{a+n^*}}{\Gamma(a+n^*)}\, \lambda_2^{a+n^*-1}\, e^{-\lambda_2(d_1\alpha+d_2+b)}. \tag{4.13}$$

The proportionality constant, c_1, for the posterior distribution of α given in (4.12) can be found using numerical techniques. However, generation from (4.12) is not a trivial issue. Hence, we propose to use the importance sampling (see Algorithm 4.4.1) to compute the Bayes estimate and as well as to construct CRI of $g(\alpha, \lambda_2)$ noting the following representation of $l_2(\alpha,\, \lambda_2 \,|\, \text{Data})$. For $0 < \alpha < 1$ and $\lambda_2 > 0$

$$l_2(\alpha,\, \lambda_2 \,|\, \text{Data}) = c_1 w_1(\alpha) \times l_4(\lambda_2 \,|\, \alpha,\, \text{Data}), \tag{4.14}$$

where

$$w_1(\alpha) = \frac{\alpha^{n_1^*+c-1}(1-\alpha)^{d-1}}{(d_1\alpha + d_2 + b)^{a+n^*}}. \tag{4.15}$$

Algorithm 4.4.1

Step 1. Generate α_1 from U(0, 1) distribution.

Step 2. For the given α_1, generate λ_{21} from (4.13).

Step 3. Continue the process M times to get $\{(\alpha_1,\, \lambda_{21}),\, \ldots,\, (\alpha_M,\, \lambda_{2M})\}$.

Step 4. Compute $g_i = g(\alpha_i,\, \lambda_{2i});\ i = 1(1)M$.

Step 5. Calculate the weights $w_{1i} = c_1 w_1(\alpha_i);\ i = 1(1)M$.

Step 6. Compute the Bayes estimate of $g(\alpha, \lambda_2)$ as

$$\widehat{g}(\alpha, \lambda_2) = \frac{1}{M}\sum_{j=1}^{M} w_{1j} g_j.$$

Step 7. To construct a $100(1-\gamma)\%$, $0 < \gamma < 1$, CRI of $g(\alpha,\, \lambda_2)$, first order g_j for $j = 1,\, \ldots,\, M$, say $g_{(1)} < g_{(2)} < \ldots < g_{(M)}$, and order w_{1j} accordingly to get $w_{1(1)},\, w_{1(2)},\, \ldots,\, w_{1(M)}$. Note that $w_{1(1)},\, w_{1(2)},\, \ldots,\, w_{1(M)}$ may not be ordered. A $100(1-\gamma)\%$ CRI can be obtained as $(g_{(j_1)}, g_{(j_2)})$, where j_1 and j_2 satisfy

$$j_1,\, j_2 \in \{1,\, 2,\, \ldots,\, M\}, \quad j_1 < j_2, \quad \sum_{i=j_1}^{j_2} w_{1(i)} \le 1-\gamma < \sum_{i=j_1}^{j_2+1} w_{1(i)}. \tag{4.16}$$

The $100(1-\gamma)\%$ HPD CRI of $g(\alpha,\, \lambda_2)$ becomes $(g_{(j_1^*)},\, g_{(j_2^*)})$, where $j_1^* < j_2^*$ satisfy

$$j_1^*,\, j_2^* \in \{1,\, 2,\, \ldots,\, M\}, \quad \sum_{i=j_1^*}^{j_2^*} w_{1(i)} \le 1-\gamma < \sum_{i=j_1^*}^{j_2^*+1} w_{1(i)}, \quad g_{(j_2^*)} - g_{(j_1^*)} \le g_{(j_2)} - g_{(j_1)},$$

for all j_1 and j_2 satisfying (4.16).

Next we consider HPD credible set for $(\alpha,\, \lambda_2)$. Note that a subset $\mathcal{C}$ of $\mathbb{R}^2$ is said to be a $100(1-\gamma)\%$, $0 < \gamma < 1$, HPD credible set for $(\alpha,\, \lambda_2)$ if

$$\mathcal{C}_\gamma = \left\{(\alpha, \lambda_2) \in \mathbb{R}^2 : l(\alpha,\, \lambda_2 \,|\, \text{Data}) \ge c_\gamma\right\},$$

where c_γ is such that

$$\iint_{\mathcal{C}_\gamma} l(\alpha,\, \lambda_2 \,|\, \text{Data}) d\lambda_2\, d\alpha = \gamma \iint_{\mathbb{R}^2} l(\alpha,\, \lambda_2 \,|\, \text{Data}) d\lambda_2\, d\alpha.$$

However, close form of the set $\mathcal{C}$ cannot be obtained, as the integration of the function $l(\alpha,\, \lambda_2 \,|\, \text{Data})$ is not possible analytically. Hence, we suggest the following algorithm to construct $100(1-\gamma)\%$ HPD credible set for $(\alpha,\, \lambda_2)$.

Algorithm 4.4.2

Step 1. Follow the first 5 steps of Algorithm 4.4.1.

Step 2. Arrange $\{(\alpha_1,\, \lambda_{21},\, w_{11}),\, \ldots,\, (\alpha_M,\, \lambda_{2M},\, w_{1M})\}$ according to the descending magnitude of the function $l(\alpha,\, \lambda_2 \,|\, \text{Data})$ at those points to get $\{(\widetilde{\alpha}_i,\, \widetilde{\lambda}_{2i},\, \widetilde{w}_i),\, \ldots,\, (\widetilde{\alpha}_M,\, \widetilde{\lambda}_{2M},\, \widetilde{w}_M)\}$.

Step 3. Find the integer M_γ such that

$$\sum_{i=1}^{M_\gamma} \widetilde{w}_i \le \gamma < \sum_{i=1}^{M_\gamma+1} \widetilde{w}_i.$$

Step 4. Construct the HPD credible set for $(\alpha,\, \lambda_2)$ as

$$\mathcal{C} = \left\{(\alpha,\, \lambda_2) : l(\alpha,\, \lambda_2 \,|\, \text{Data}) \ge l(\widetilde{\alpha}_{M_\gamma},\, \widetilde{\lambda}_{2M_\gamma} \,|\, \text{Data})\right\}.$$

Similar methodology can be applied for other censoring schemes, and we will briefly mention all the cases in Section 4.6 for completeness purposes.

4.5 Simulations and Data Analysis

4.5.1 Simulation Results

In this section we present some simulation results to see how the BE works for different sample sizes and for different values of τ_1 and τ_2. Along with the CP and AL of symmetric CRI, HPD CRI, same of bootstrap CI is also presented for a comparison purpose. Here we choose $\lambda_1 = 1/12 \simeq 0.083$ and $\lambda_2 = 1/4.5 \simeq 0.222$. We also choose $a = 0$, $b = 0$, $c = 1$ and $d = 1$, *i.e.*, the non-informative prior and hence, the comparison with MLE is meaningful. All the results are based on 5000

Table 4.1: AE and MSE of MLE and BE of λ_1 based on 5000 simulations with $\lambda_1 = 0.083$, $\lambda_2 = 0.222$, $a = 0$, $b = 0$, $c = 1$, and $d = 1$ for the Type-I censored case.

			λ_1				λ_2			
			BE		MLE		BE		MLE	
n	τ_1	τ_2	AE	MSE	AE	MSE	AE	MSE	AE	MSE
10	5	6	0.090	0.0018	0.082	0.0019	0.267	0.0223	0.321	0.0499
		8	0.092	0.0017	0.087	0.0021	0.236	0.0154	0.256	0.0290
		10	0.094	0.0017	0.087	0.0022	0.238	0.0136	0.256	0.0230
10	7	8	0.087	0.0014	0.081	0.0015	0.283	0.0369	0.359	0.1009
		10	0.090	0.0015	0.086	0.0018	0.243	0.0189	0.270	0.0454
		12	0.091	0.0015	0.087	0.0019	0.241	0.0164	0.264	0.0350
10	9	10	0.085	0.0012	0.079	0.0012	0.301	0.0409	0.411	0.1463
		12	0.088	0.0012	0.084	0.0014	0.257	0.0538	0.310	0.4920
		14	0.089	0.0013	0.085	0.0015	0.251	0.0271	0.288	0.0916
20	5	6	0.086	0.0009	0.083	0.0010	0.227	0.0111	0.248	0.0210
		8	0.088	0.0009	0.084	0.0011	0.221	0.0065	0.235	0.0096
		10	0.090	0.0009	0.084	0.0010	0.221	0.0053	0.234	0.0073
20	7	8	0.086	0.0008	0.083	0.0009	0.230	0.0116	0.255	0.0239
		10	0.088	0.0007	0.085	0.0008	0.219	0.0070	0.235	0.0115
		12	0.088	0.0007	0.084	0.0008	0.222	0.0059	0.238	0.0088
20	9	10	0.084	0.0006	0.082	0.0006	0.240	0.0142	0.275	0.0319
		12	0.087	0.0006	0.084	0.0007	0.224	0.0090	0.243	0.0156
		14	0.087	0.0006	0.084	0.0007	0.225	0.0078	0.242	0.0125
30	5	6	0.084	0.0006	0.082	0.0007	0.216	0.0079	0.233	0.0139
		8	0.088	0.0006	0.083	0.0007	0.217	0.0044	0.230	0.0059
		10	0.089	0.0007	0.083	0.0007	0.220	0.0034	0.231	0.0043
30	7	8	0.084	0.0005	0.083	0.0005	0.217	0.0077	0.234	0.0148
		10	0.087	0.0005	0.083	0.0005	0.216	0.0048	0.230	0.0067
		12	0.088	0.0005	0.084	0.0005	0.218	0.0041	0.231	0.0054
30	9	10	0.084	0.0004	0.082	0.0004	0.223	0.0091	0.244	0.0185
		12	0.086	0.0004	0.084	0.0004	0.219	0.0058	0.235	0.0089
		14	0.086	0.0004	0.083	0.0005	0.217	0.0049	0.232	0.0066
40	5	6	0.085	0.0005	0.083	0.0005	0.213	0.0061	0.229	0.0099
		8	0.087	0.0005	0.083	0.0005	0.218	0.0035	0.230	0.0043
		10	0.088	0.0005	0.083	0.0005	0.218	0.0026	0.227	0.0031
40	7	8	0.085	0.0004	0.083	0.0004	0.214	0.0067	0.231	0.0118
		10	0.086	0.0004	0.083	0.0004	0.214	0.0039	0.228	0.0050
		12	0.086	0.0004	0.082	0.0004	0.218	0.0033	0.230	0.0040
40	9	10	0.084	0.0003	0.082	0.0003	0.217	0.0078	0.235	0.0145
		12	0.085	0.0003	0.083	0.0003	0.217	0.0048	0.232	0.0065
		14	0.085	0.0003	0.083	0.0003	0.217	0.0038	0.230	0.0048

simulations and $M = 8000$. We chose $n = 10$ (small sample size), 20 (moderate sample size), 30, and 40 (large sample size). AE and MSE of BE along with that of MLE for λ_1 and λ_2 are presented in the Table 4.1 for different values of τ_1 and τ_2. The CP and AL of symmetric CRI, HPD CRI, and bootstrap CI for same n, τ_1, and τ_2 are reported in the Tables 4.2, 4.3, and 4.4. We also report the CP of

Table 4.2: CP and AL of 90% CRIs and CI for λ_1 and λ_2 based on 5000 simulations with $\lambda_1 = 0.083$, $\lambda_2 = 0.222$, $a = 0$, $b = 0$, $c = 1$, and $d = 1$ for the Type-I censored case.

			λ_1						λ_2					
			Symm. CRI		HPD CRI		Boot. CI		Symm. CRI		HPD CRI		Boot. CI	
n	τ_1	τ_2	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL		
10	5	6	92.58	0.140	88.48	0.132	86.92	0.123	97.96	0.545	95.74	0.469	98.62	0.679
		8	93.92	0.136	89.64	0.128	89.62	0.125	91.56	0.361	85.96	0.332	91.96	0.458
		10	94.26	0.137	89.90	0.130	89.40	0.128	90.12	0.328	86.00	0.307	91.02	0.431
10	7	8	89.80	0.124	87.18	0.118	84.54	0.111	97.12	0.608	94.48	0.512	98.18	0.875
		10	91.84	0.123	88.88	0.117	86.20	0.117	92.16	0.409	88.76	0.365	93.04	0.605
		12	92.22	0.123	90.46	0.118	86.18	0.119	91.12	0.364	87.66	0.333	92.08	0.543
10	9	10	90.32	0.114	87.20	0.109	81.88	0.103	98.16	0.687	97.66	0.567	97.30	1.139
		12	91.80	0.114	90.02	0.109	84.34	0.109	93.84	0.450	88.26	0.395	94.90	0.766
		14	92.30	0.115	91.62	0.111	86.02	0.112	91.62	0.415	87.24	0.372	92.48	0.702
20	5	6	90.14	0.100	89.02	0.096	84.12	0.098	94.94	0.359	88.80	0.324	92.58	0.407
		8	92.04	0.100	91.64	0.097	85.96	0.099	90.36	0.256	85.46	0.243	89.64	0.291
		10	91.58	0.100	91.38	0.097	84.82	0.098	89.78	0.229	86.40	0.221	90.12	0.261
20	7	8	89.64	0.089	87.86	0.087	88.76	0.089	95.64	0.388	89.84	0.344	95.92	0.452
		10	92.22	0.089	91.26	0.087	91.12	0.090	90.18	0.278	85.74	0.261	89.34	0.325
		12	92.34	0.090	91.16	0.088	90.54	0.090	90.08	0.249	86.70	0.238	90.00	0.293
20	9	10	90.60	0.082	88.90	0.080	88.52	0.082	96.94	0.423	92.34	0.368	98.18	0.513
		12	91.46	0.083	90.84	0.081	89.30	0.084	90.78	0.298	85.94	0.276	89.96	0.360
		14	92.40	0.083	91.72	0.081	89.88	0.084	90.30	0.270	86.20	0.255	89.72	0.331
30	5	6	90.42	0.083	88.16	0.081	85.48	0.083	91.40	0.301	85.98	0.277	88.70	0.336
		8	91.98	0.083	90.52	0.081	87.38	0.082	89.24	0.215	85.62	0.207	88.46	0.236
		10	91.12	0.084	89.76	0.082	87.18	0.083	89.34	0.191	86.52	0.186	89.84	0.209
30	7	8	90.40	0.074	89.64	0.072	87.96	0.074	93.28	0.320	86.54	0.291	91.70	0.366
		10	90.52	0.074	90.10	0.073	88.14	0.074	88.96	0.231	84.60	0.220	88.08	0.258
		12	90.24	0.074	90.28	0.073	88.00	0.075	89.76	0.206	86.90	0.199	90.00	0.230
30	9	10	90.46	0.068	89.48	0.067	88.46	0.068	95.00	0.344	88.76	0.308	92.94	0.402
		12	90.64	0.068	90.48	0.067	88.74	0.069	89.70	0.249	84.24	0.236	88.02	0.284
		14	90.12	0.068	90.06	0.067	88.14	0.069	89.20	0.222	85.08	0.213	89.14	0.254
40	5	6	90.50	0.072	89.58	0.071	87.66	0.072	89.80	0.266	84.06	0.248	87.48	0.292
		8	91.30	0.073	90.72	0.072	87.82	0.073	88.72	0.191	85.16	0.185	88.24	0.205
		10	90.70	0.074	90.92	0.072	87.84	0.073	89.38	0.167	86.38	0.163	89.46	0.179
40	7	8	90.80	0.064	90.62	0.063	89.10	0.065	91.22	0.281	84.08	0.259	88.04	0.312
		10	89.70	0.065	89.34	0.064	87.26	0.065	89.42	0.205	85.02	0.197	88.46	0.223
		12	90.76	0.065	90.72	0.064	88.46	0.065	89.44	0.181	86.18	0.177	89.66	0.197
40	9	10	90.80	0.059	89.80	0.058	89.00	0.059	91.86	0.301	84.98	0.273	89.42	0.340
		12	90.82	0.059	90.70	0.058	89.16	0.060	88.56	0.218	83.82	0.209	87.30	0.242
		14	90.66	0.060	90.82	0.059	88.92	0.060	88.80	0.195	85.42	0.189	88.62	0.215

the HPD credible set for $(\alpha,\, \lambda_2)$ in the Table 4.5 using Algorithm 4.4.2 for the same parametric values.

The following points are quite clear from the simulation results. MSE of estimator of λ_1 decreases as τ_1 increases when other parameters are held constant. MSE of estimator of λ_2 decreases as τ_2 increases, whereas it increases as τ_1 increases. Also

Table 4.3: CP and AL of 95% CRIs and CI for λ_1 and λ_2 based on 5000 simulations with $\lambda_1 = 0.083$, $\lambda_2 = 0.222$, $a = 0$, $b = 0$, $c = 1$, and $d = 1$ for the Type-I censored case.

			λ_1						λ_2					
			Symm. CRI		HPD CRI		Boot. CI		Symm. CRI		HPD CRI		Boot. CI	
n	τ_1	τ_2	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL		
10	5	6	96.40	0.169	95.46	0.158	89.64	0.147	99.22	0.680	99.44	0.594	99.42	0.877
		8	97.52	0.163	96.52	0.154	91.10	0.150	96.12	0.439	93.82	0.406	96.28	0.592
		10	97.70	0.164	97.44	0.156	91.30	0.154	95.24	0.397	93.32	0.372	95.40	0.567
10	7	8	95.70	0.149	92.56	0.141	86.48	0.134	99.20	0.766	98.72	0.658	99.22	1.203
		10	95.74	0.147	94.68	0.141	87.80	0.141	96.90	0.502	94.12	0.455	97.82	0.844
		12	96.14	0.148	95.60	0.141	88.26	0.143	95.60	0.443	93.20	0.409	96.54	0.767
10	9	10	95.52	0.137	93.46	0.131	89.06	0.123	99.34	0.871	99.58	0.737	98.52	1.670
		12	96.26	0.136	94.64	0.131	92.92	0.131	97.36	0.556	95.62	0.496	97.96	1.151
		14	96.74	0.138	95.34	0.133	93.94	0.136	95.82	0.510	94.32	0.461	96.60	1.056
20	5	6	95.36	0.119	93.56	0.115	93.30	0.118	97.96	0.440	96.14	0.400	98.12	0.493
		8	96.32	0.120	96.10	0.116	94.08	0.119	95.60	0.308	93.02	0.293	95.18	0.354
		10	96.48	0.120	96.10	0.116	93.48	0.118	95.18	0.275	93.22	0.265	95.10	0.320
20	7	8	94.36	0.106	93.74	0.103	92.76	0.107	98.36	0.478	97.00	0.429	99.02	0.553
		10	96.18	0.107	96.06	0.104	94.78	0.108	95.34	0.336	92.82	0.317	95.16	0.399
		12	96.22	0.107	96.18	0.105	94.30	0.109	95.38	0.299	93.14	0.286	95.32	0.363
20	9	10	95.38	0.098	94.52	0.096	94.36	0.098	99.34	0.524	97.92	0.464	99.62	0.636
		12	95.96	0.099	95.28	0.097	94.72	0.101	95.72	0.361	92.96	0.336	95.84	0.450
		14	96.30	0.099	95.78	0.096	95.16	0.101	95.34	0.325	93.22	0.308	95.74	0.419
30	5	6	95.12	0.099	94.36	0.097	91.02	0.099	97.04	0.365	93.60	0.338	95.90	0.403
		8	96.04	0.099	95.36	0.097	92.08	0.099	94.90	0.258	92.62	0.248	94.66	0.284
		10	95.82	0.100	95.08	0.098	92.02	0.099	94.84	0.228	92.68	0.222	94.96	0.253
30	7	8	95.08	0.088	94.66	0.086	93.94	0.089	97.68	0.390	95.20	0.357	97.06	0.440
		10	95.50	0.088	95.16	0.087	93.80	0.089	94.90	0.277	91.94	0.265	94.10	0.312
		12	95.50	0.089	95.20	0.087	93.84	0.089	94.92	0.247	93.26	0.239	94.68	0.279
30	9	10	95.08	0.081	94.70	0.080	94.16	0.082	98.72	0.421	96.30	0.381	97.84	0.486
		12	95.48	0.082	95.34	0.080	94.30	0.083	94.90	0.299	92.40	0.284	94.32	0.346
		14	95.36	0.081	94.98	0.080	93.68	0.083	94.00	0.266	91.78	0.256	94.46	0.311
40	5	6	95.24	0.086	94.74	0.085	94.02	0.086	95.58	0.321	92.22	0.301	93.92	0.349
		8	95.74	0.087	95.44	0.086	94.44	0.087	94.02	0.228	91.76	0.221	93.46	0.246
		10	95.60	0.088	95.52	0.086	94.32	0.087	94.36	0.199	92.68	0.195	94.78	0.215
40	7	9	95.46	0.077	95.48	0.075	94.98	0.077	96.42	0.340	93.06	0.315	94.92	0.374
		10	94.98	0.077	94.66	0.076	94.02	0.078	94.72	0.245	92.48	0.236	94.40	0.269
		12	95.22	0.077	95.34	0.076	94.74	0.078	94.76	0.217	93.00	0.211	94.72	0.238
40	9	10	95.22	0.070	94.78	0.069	94.16	0.071	97.06	0.366	93.80	0.335	96.52	0.409
		12	95.38	0.071	94.92	0.070	94.14	0.071	94.42	0.261	91.38	0.251	93.66	0.292
		14	95.16	0.071	95.56	0.070	94.64	0.072	94.34	0.233	92.02	0.225	94.50	0.261

MSEs of all the estimators decrease as n increases. MSE of MLE is close to that of BE for λ_1, but MSE of MLE is larger than MSE of BE for λ_2. This difference decreases as τ_2 increases.

The performance of CI and CRIs of λ_1 are quite satisfactory for all the sample sizes. We note that average length of HPD CRI of λ_1 decreases as τ_1 or n increases,

Table 4.4: CP and AL of 99% CRIs and CI for λ_1 and λ_2 based on 5000 simulations with $\lambda_1 = 0.083$, $\lambda_2 = 0.222$, $a = 0$, $b = 0$, $c = 1$, and $d = 1$ for the Type-I censored case.

			λ_1						λ_2					
			Symm. CRI		HPD CRI		Boot. CI		Symm. CRI		HPD CRI		Boot. CI	
n	τ_1	τ_2	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL		
10	5	6	99.76	0.226	99.90	0.214	99.94	0.201	99.86	0.985	99.96	0.883	99.88	1.548
		8	99.72	0.218	99.82	0.207	99.94	0.205	99.60	0.606	99.00	0.565	99.56	1.044
		10	99.58	0.218	99.70	0.208	99.72	0.210	98.96	0.540	98.60	0.509	98.80	1.046
10	7	8	99.36	0.199	98.44	0.190	96.72	0.182	99.88	1.127	99.98	1.001	99.58	2.749
		10	99.44	0.196	99.40	0.187	97.30	0.191	99.52	0.703	99.36	0.646	99.58	2.097
		12	99.50	0.196	99.54	0.188	97.32	0.195	99.24	0.613	98.64	0.571	99.22	1.880
10	9	10	99.24	0.182	98.78	0.174	96.10	0.168	99.90	1.297	99.94	1.142	99.22	4.755
		12	99.32	0.181	98.96	0.174	97.50	0.178	99.86	0.802	99.52	0.724	99.42	3.422
		14	99.36	0.184	99.08	0.177	97.82	0.184	99.36	0.711	98.82	0.656	99.28	3.110
20	5	6	99.00	0.159	98.66	0.153	98.00	0.157	99.92	0.615	99.78	0.568	99.96	0.680
		8	99.38	0.158	99.48	0.153	98.52	0.158	99.32	0.415	98.92	0.396	99.22	0.495
		10	99.34	0.158	99.44	0.153	98.38	0.157	99.00	0.367	98.74	0.354	98.96	0.454
20	7	8	98.94	0.141	98.46	0.137	97.10	0.143	99.90	0.675	99.82	0.618	99.96	0.789
		10	99.28	0.141	99.08	0.137	97.92	0.145	99.32	0.454	98.62	0.431	99.22	0.578
		12	99.32	0.141	99.20	0.138	97.84	0.146	99.12	0.401	98.68	0.384	99.16	0.536
20	9	10	99.04	0.130	98.60	0.127	98.26	0.132	99.90	0.748	99.94	0.679	99.94	0.960
		12	99.14	0.131	99.02	0.127	98.18	0.136	99.50	0.492	98.76	0.462	99.40	0.698
		14	99.14	0.130	98.96	0.127	98.48	0.135	99.32	0.438	98.84	0.416	99.42	0.673
30	5	6	99.24	0.131	98.70	0.127	97.18	0.131	99.76	0.500	99.58	0.468	99.58	0.541
		8	99.20	0.131	98.98	0.128	97.20	0.131	98.84	0.343	98.38	0.331	98.70	0.385
		10	99.36	0.132	99.20	0.129	97.52	0.132	98.82	0.302	98.32	0.294	98.98	0.346
30	7	8	99.00	0.116	98.80	0.114	98.08	0.118	99.86	0.540	99.80	0.502	99.78	0.598
		10	99.02	0.116	99.06	0.114	98.14	0.119	99.08	0.369	98.40	0.355	98.96	0.428
		12	99.14	0.117	99.02	0.115	98.22	0.119	99.06	0.327	98.38	0.317	99.14	0.388
30	9	10	99.06	0.107	98.82	0.105	98.30	0.109	99.88	0.587	99.88	0.541	99.88	0.671
		12	99.16	0.107	99.02	0.105	98.58	0.110	99.44	0.401	98.80	0.383	99.30	0.484
		14	99.18	0.107	99.10	0.105	98.42	0.110	98.90	0.354	98.40	0.341	98.94	0.441
40	5	6	99.14	0.114	98.86	0.112	98.12	0.114	99.60	0.434	98.84	0.411	98.96	0.462
		8	99.28	0.115	99.00	0.113	98.44	0.115	98.70	0.301	97.90	0.293	98.38	0.329
		10	99.06	0.115	98.88	0.113	98.32	0.115	98.84	0.263	98.34	0.257	98.80	0.290
40	7	8	99.24	0.101	99.04	0.099	98.78	0.103	99.80	0.464	99.32	0.435	99.64	0.500
		10	99.12	0.102	99.04	0.100	98.64	0.103	99.00	0.325	98.42	0.314	98.80	0.361
		12	99.26	0.102	99.12	0.100	98.76	0.103	98.98	0.286	98.46	0.278	98.84	0.324
40	9	10	99.08	0.093	98.90	0.091	98.68	0.094	99.80	0.502	99.60	0.468	99.70	0.554
		12	98.94	0.093	98.78	0.091	98.42	0.095	99.28	0.348	98.36	0.334	99.02	0.397
		14	99.10	0.094	99.08	0.092	98.80	0.095	99.08	0.308	98.30	0.298	99.00	0.359

keeping the other fixed. HPD CRI of λ_1 performs quite well compared to the symmetric CRI and bootstrap CI with respect to CP of the respective intervals, though AL of HPD CRI is larger than that of the bootstrap CI, but smaller than that of symmetric CRI. For moderate and large sample sizes, AL of symmetric CRI, HPD CRI and bootstrap CI are very close to each other.

Table 4.5: CP of credible set for $(\alpha,\, \lambda_2)$ based on 5000 simulations with $\lambda_1 = 0.083$, $\lambda_2 = 0.222$, $a = 0$, $b = 0$, $c = 1$, and $d = 1$ for the Type-I censored case.

	$\tau_1 = 5$				$\tau_1 = 7$				$\tau_1 = 9$			
		CP				CP				CP		
n	τ_2	90%	95%	99%	τ_2	90%	95%	99%	τ_2	90%	95%	99%
10	6	87.26	91.20	98.16	8	86.90	91.56	97.30	10	85.70	94.40	98.92
	8	85.26	91.08	97.98	10	84.08	92.76	98.50	12	87.08	92.86	98.04
	10	85.20	92.12	98.54	12	85.34	94.02	98.32	14	86.50	92.76	98.20
20	6	84.84	92.66	98.30	8	87.02	93.48	98.54	10	86.84	93.70	98.72
	8	85.44	92.38	98.28	10	84.36	93.00	98.80	12	85.34	93.08	98.82
	10	86.48	92.56	98.56	12	85.50	92.56	98.36	14	85.20	92.68	98.78
30	6	84.30	92.28	98.26	8	85.52	93.54	98.82	10	84.84	93.66	98.70
	8	85.14	92.00	98.24	10	85.54	92.76	98.50	12	85.74	93.00	98.68
	10	85.54	92.70	98.48	12	86.12	92.76	98.46	14	86.32	93.10	98.42
40	6	84.42	91.76	98.30	8	83.34	91.96	98.60	10	85.94	93.60	98.86
	8	85.74	91.84	98.36	10	84.96	91.94	98.26	12	85.30	92.26	98.38
	10	86.72	93.12	98.68	12	86.70	93.14	98.66	14	85.56	92.46	98.40

The performance of the HPD CRI is not so satisfactory for λ_2 with respect to CP. However, CP of symmetric CRI and bootstrap CI is close to nominal level when $(\tau_2 - \tau_1)$ is large. For small sample size performance of CRIs as well as bootstrap CI for λ_2 not at all satisfactory, specially when $(\tau_2 - \tau_1)$ is small. ALs of CRIs and bootstrap CI of λ_2 decrease as τ_2 or n increases, keeping the other parameter constant. Also note that ALs of bootstrap CI and HPD CRI of the same parameter increase as τ_1 increases.

4.5.2 Data Analysis

Example 1

Here we consider the data (see Table 4.6) presented by Xiong [136] to illustrate the methods of estimation discussed previously. This is actually a Type-II censored data from a simple step stress life experiment, where $n = 20$ units are placed on the test, the data is right censored at 16th failure time and stress changing time is $\tau_1 = 5$. Balakrishnan et al. [36] used this data for illustrative example for step-stress model under Type-I censoring scheme choosing $\tau_2 = 7$, 8, 9, and 12. They reported the

Table 4.6: Data of Example 1.

Stress Level	Failure Times					
$\lambda_1 = e^{-2.5}$	2.01	3.60	4.12	4.34		
$\lambda_2 = e^{-1.5}$	5.04	5.94	6.68	7.09	7.17	7.49
	7.60	8.23	8.24	8.25	8.69	12.05

Table 4.7: Estimates of $1/\lambda_1$ and associated CRIs and bootstrap CI for the data in Table 4.6.

				Symm. CRI		HPD CRI		Boot. CI	
Level	τ_2	BE	MLE	LL	UL	LL	UL	LL	UL
90%	7	27.495	23.517	12.232	54.633	9.628	45.987	10.844	49.290
	8	23.933	23.517	10.925	48.281	8.625	40.377	10.602	49.286
	9	23.367	23.517	10.389	47.056	7.695	38.585	10.352	49.108
	12	23.600	23.517	10.528	46.885	7.878	39.072	10.352	49.106
95%	7	-	-	11.007	64.662	9.066	55.794	10.983	98.112
	8	-	-	9.682	58.370	7.304	49.108	10.854	98.272
	9	-	-	9.232	57.491	7.306	47.484	10.879	98.395
	12	-	-	9.477	57.757	7.517	47.627	10.854	98.336
99%	7	-	-	9.022	97.388	7.297	82.415	9.655	99.983
	8	-	-	7.785	84.090	6.032	73.248	9.507	99.991
	9	-	-	7.632	82.255	6.156	71.762	9.425	99.990
	12	-	-	7.837	86.340	6.275	73.292	9.425	99.990

Table 4.8: Estimates of $1/\lambda_2$ and associated CRIs and bootstrap CI with $a = 0$, $b = 0$, $c = 1$, $d = 1$ for the data in Table 4.6

				Symm. CRI		HPD CRI		Boot. CI	
Level	τ_2	BE	MLE	LL	UL	LL	UL	LL	UL
90%	7	13.285	9.553	5.356	26.544	3.712	22.596	3.245	18.778
	8	7.426	5.573	3.649	13.735	3.024	11.965	2.391	10.097
	9	5.055	4.129	2.895	8.425	2.462	7.478	2.034	6.468
	12	6.684	5.493	3.837	11.052	3.271	9.853	2.966	8.400
95%	7	-	-	4.605	31.846	3.343	26.910	2.718	25.117
	8	-	-	3.293	15.705	2.737	13.918	2.315	12.330
	9	-	-	2.680	9.527	2.167	8.506	2.025	7.613
	12	-	-	3.502	12.297	3.109	11.322	2.703	9.398
99%	7	-	-	3.635	46.657	2.625	38.544	2.718	35.850
	8	-	-	2.745	20.340	2.273	18.528	1.908	17.677
	9	-	-	2.200	12.091	2.035	11.120	1.531	10.028
	12	-	-	2.973	15.747	2.685	14.395	2.164	11.752

Table 4.9: Credible set for (α, λ_2) with $a = 0$, $b = 0$, $c = 1$, and $d = 1$ for data in Table 4.6.

τ_2	n_2	d_2	c_0	$c_{0.90}$	$c_{0.95}$	$c_{0.99}$
7	3	28.66	3.95873×10^{10}	0.000345079	0.000185004	0.000042293
8	7	39.01	4.26825×10^{18}	0.000302864	0.000170350	0.000035177
9	11	54.42	3.43915×10^{26}	0.000275705	0.000127509	0.000032715
2	11	60.42	6.06288×10^{27}	0.000310440	0.000167296	0.000048826

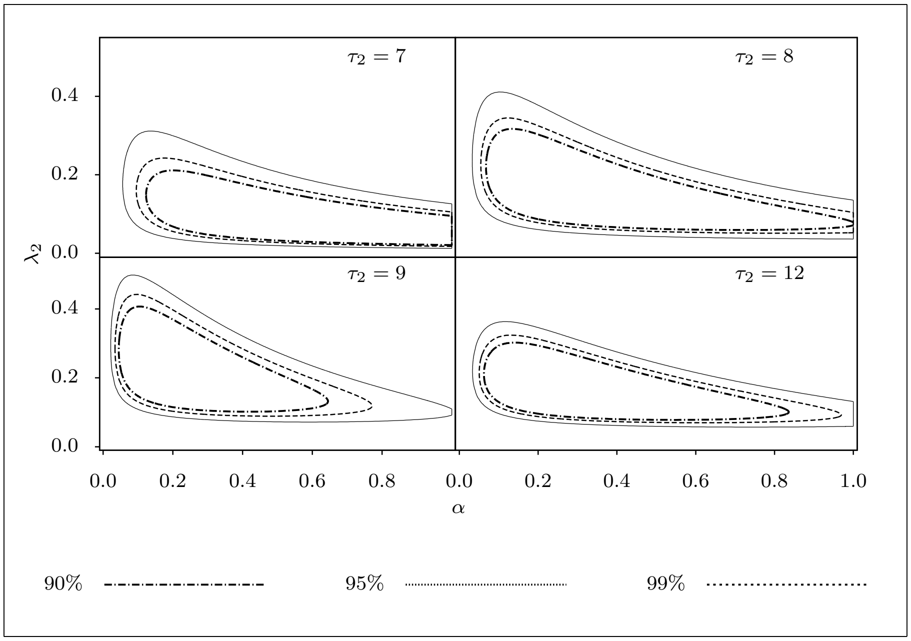


Figure 4.2: Credible set of (α, λ_2) with $a = 0$, $b = 0$, $c = 1$, and $d = 1$ for the data in Table 4.6.

MLE of $1/\lambda_1$ and $1/\lambda_2$ and associated CIs for above mentioned τ_1 and τ_2's under the assumption that the data are coming form exponential CEM. The BE and Bayesian CRIs for $1/\lambda_1$ and $1/\lambda_2$ are reported in Tables 4.7 and 4.8, respectively for the same values of parameters and under the same assumption. Here we choose $a = 0$, $b = 0$, $c = 1$, $d = 1$ and $M = 8000$. We also find out HPD credible set for (α, λ_2) and is given by

$$\mathcal{C}_\gamma = \left\{ (\alpha, \lambda_2) \in \mathbb{R}^2 : \frac{c_0}{\Gamma(n_2 + 4)} \alpha^4 \lambda_2^{n_2+3} e^{94.07\alpha + d_2} \geq c_\gamma \right\},$$

where c_0, n_2, and d_2 depend on τ_2 and are presented in Table 4.9. Figure 4.2 shows the plot of the HPD credible set of (α, λ_2) for different values of τ_2.

Example 2

Next, we consider the data (see Table 4.10) presented by Balakrishnan et al. [36]. Choice made by Balakrishnan et al. [36] were $n = 35$, $\lambda_1 = e^{-3.5}$, $\lambda_2 = e^{-2.0}$ and

Table 4.10: Data of Example 2.

Stress Level	Failure Times								
$\lambda_1 = e^{-3.5}$	1.46	2.22	3.92	4.24	5.47	5.60	6.12	6.57	
$\lambda_2 = e^{-2.0}$	8.19	8.30	8.74	8.98	9.43	9.87	11.14	11.76	11.85
	12.14	13.05	13.49	14.04	14.19	14.24	14.33	15.28	16.58
	16.85	16.92	17.80	20.45	20.98	21.09	22.01	26.34	28.66

Table 4.11: Estimates of $1/\lambda_1$ and associated CRIs and bootstrap CI with $a = 0$, $b = 0$, $c = 1$, $d = 1$ for the data in Table 4.10

				Symm. CRI		HPD CRI		Boot. CI	
Level	τ_2	BE	MLE	LL	UL	LL	UL	LL	UL
90%	12	31.967	31.450	17.991	53.803	15.222	47.454	16.733	53.478
	16	31.613	31.450	17.367	53.615	15.116	47.665	16.431	53.524
	20	31.487	31.450	17.566	54.091	14.503	47.439	16.392	53.596
	24	31.260	31.450	17.285	53.291	14.810	47.399	16.667	53.328
95%	12	-	-	16.531	61.203	14.826	55.588	16.144	66.821
	16	-	-	16.078	61.470	14.121	55.428	16.450	67.424
	20	-	-	16.012	60.247	13.845	55.387	16.392	67.377
	24	-	-	15.801	60.956	13.762	53.914	16.591	67.321
99%	12	-	-	13.954	79.227	12.098	72.962	13.651	91.571
	16	-	-	13.568	81.695	11.584	73.471	13.814	91.436
	20	-	-	13.715	77.763	11.791	70.883	12.953	91.560
	24	-	-	13.720	81.838	11.594	73.771	13.551	91.087

Table 4.12: Estimates of $1/\lambda_2$ and associated CRIs and bootstrap CI with $a = 0$, $b = 0$, $c = 1$, $d = 1$ for the data in Table 4.10

				Symm. CRI		HPD CRI		Boot. CI	
Level	τ_2	BE	MLE	LL	UL	LL	UL	LL	UL
90%	12	12.225	9.807	6.628	20.765	5.717	18.738	5.135	16.472
	16	9.548	8.413	6.160	14.233	5.707	13.287	5.321	12.149
	20	9.021	8.151	6.143	13.031	5.877	12.451	5.337	11.255
	24	7.979	7.348	5.605	11.139	5.208	10.444	4.867	9.860
95%	12	-	-	6.041	23.307	5.091	21.052	4.517	18.763
	16	-	-	5.767	15.699	5.261	14.510	4.675	13.169
	20	-	-	5.767	14.095	5.312	13.223	4.851	12.109
	24	-	-	5.289	11.942	5.025	11.355	4.526	10.556
99%	12	-	-	5.217	29.326	4.899	27.208	3.966	25.171
	16	-	-	5.136	18.469	4.796	17.246	4.178	16.308
	20	-	-	5.143	16.804	4.921	15.681	4.274	14.424
	24	-	-	4.933	13.693	4.726	13.009	4.088	12.253

Table 4.13: Credible set for (α, λ_2) with $a = 0$, $b = 0$, $c = 1$, and $d = 1$ for data in Table 4.10.

τ_2	n_2	d_2	c_0	$c_{0.90}$	$c_{0.95}$	$c_{0.99}$
12	9	88.26	1.98906×10^{38}	0.000634529	0.000363521	0.000084468
16	17	143.02	1.83135×10^{59}	0.000683449	0.000368794	0.000030347
20	21	171.17	1.46811×10^{70}	0.000751840	0.000368533	0.000087536
24	25	183.70	2.77355×10^{80}	0.000804461	0.000399712	0.000078281

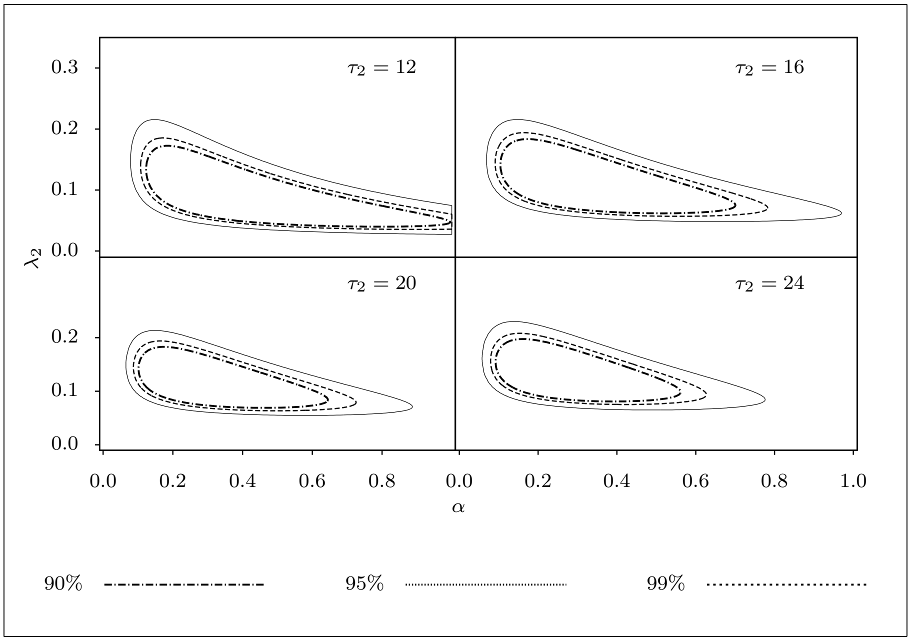


Figure 4.3: Credible set of $(\alpha,\, \lambda_2)$ with $a = 0$, $b = 0$, $c = 1$, and $d = 1$ for the data in Table 4.10.

$\tau_1 = 8$. This is a complete data set. To make it a Type-I censored data one may take any τ_2 greater than 8. Balakrishnan et al. [36] took different choices for τ_2, *viz.*, 16, 20 and 24. Along these choices of τ_2, we take $\tau_2 = 12$ also. Assuming that the data are coming form the exponential CEM under Type-I censoring, they presented MLE and associated CIs of $1/\lambda_1$ and $1/\lambda_2$. Here we present the BE of $1/\lambda_1$ and $1/\lambda_2$ and associated CRIs for the above mentioned values of τ_2 under the same assumption. The results are presented in Tables 4.11 and 4.12. Here also we choose $a = 0$, $b = 0$, $c = 1$, $d = 1$, and $M = 8000$. Like the previous example, we also find out HPD credible set for $(\alpha,\, \lambda_2)$ and is given by

$$\mathcal{C}_\gamma = \left\{ (\alpha,\, \lambda_2) \in \mathbb{R}^2 : \frac{c_0}{\Gamma(n_2+8)} \alpha^8\, \lambda_2^{n_2+7}\, e^{251.60\alpha + d_2} \geq c_\gamma \right\},$$

where c_0, n_2, and d_2 depend on τ_2 and are presented in Table 4.13. Figure 4.3 shows the plot of the HPD credible set of $(\alpha,\, \lambda_2)$ for different values of τ_2.

4.6 Posterior Analysis under Other Censoring Schemes

Type-II Censoring Scheme

Based on the observed sample, the likelihood function is given in (4.8), where $\tau^* = t_{r:n}$, in Case I, $N_1^* = 0$, $N_2^* = r$, in Case II, $N_1^* = n_1$, $N_2^* = r - n_1$ and in Case III, $N_1^* = r$, $N_2^* = 0$. d_1 and d_2 have the same expression as given in case of Type-I censoring.

Type-I Hybrid Censoring Scheme

Based on the data from Type-I HCS, the likelihood function is same as (4.8), where in Case I, $N_1^* = 0$, $N_2^* = r$, in Case II, $N_1^* = n_1$, $N_2^* = r - n_1$, in Case III, $N_1^* = r$, $N_2^* = 0$, in Case IV, $N_1^* = 0$, $N_2^* = n_2$, in Case V, $N_1^* = n_1$, $N_2^* = n_2$, and in Case VI, $N_1^* = n_1$, $N_2^* = 0$. Also in the Cases I-III, $\tau^* = t_{r:n}$, where for the rest of the cases $\tau^* = \tau_2$. d_1 and d_2 have the same expression as given in case of Type-I censoring.

Type-II Hybrid Censoring Scheme

Based on the observed sample from Type-II HCS, the likelihood function is given in (4.8), where in Case I, $N_1^* = 0$, $N_2^* = r$, for Case II, $N_1^* = n_1$, $N_2^* = r - n_1$, in Case III, $N_1^* = 0$, $N_2^* = n_2$, for Case IV, $N_1^* = n_1$, $N_2^* = n_2$ and for Case V, $N_1^* = n_1$, $N_2^* = 0$. $\tau^* = t_{r:n}$ for Cases I and II, where for the rest of the cases $\tau^* = \tau_2$. d_1 and d_2 have the same expression as given in case of Type-I censoring.

Progressive Type-II Censoring Scheme

With the observed progressive Type-II censoring data, the likelihood function is given by (4.8), where for Case I, $N_1^* = 0$, $N_2^* = m$, for Case II, $N_1^* = n_1$, $N_2^* =$

$m - n_1$ and for Case III $N_1^* = m$, $N_2^* = 0$. For all the cases $\tau^* = t_{m:n}$, $d_1 = \sum_{k=1}^{N_1^*}(R_k+1)t_{k:n} + (n - N_1^* - \sum_{k=1}^{N^*} R_k)\tau_1$ and $d_2 = \sum_{k=N_1^*+1}^{m}(R_k+1)(t_{k:n} - \tau_1)$.

In all the above cases, likelihood function are in the same form as Type-I censoring scheme and hence, the posterior density will also be in the same form as given in (4.9). In all these cases we will be able to compute the BE and construct the associated CRI for some function of α and λ_2 exactly along the same line. One can also construct credible set for α and λ_2 following the same methodology.

4.7 Conclusion

We have considered the Bayesian estimation of the unknown parameters in a simple SSLT under the restriction $\lambda_1 < \lambda_2$ and under different censoring schemes. The analysis is performed under exponentially distributed lifetimes and under CEM assumption. We have taken mainly the squared error loss function, though other loss functions can also be handled in a very similar way. We have seen that the BE of some parametric function under the square error loss function does not exist in close form in most of the cases. An algorithm based on importance sampling is proposed to compute BE and CRIs. We have done a simulation study to judge the performance of the procedures described. We also considered two data sets to illustrate the estimation procedures. We have noticed that the performance of BE and CRIs for λ_1 is quite satisfactory. It is also noticed that the performance of BE and CRI for λ_1 is better than that of MLE and bootstrap CI for small sample size and for the small values of τ_1. For moderate and large sample sizes the performance of BE and CRI is quite close to that of MLE and bootstrap CI. We have also noticed that the performance of BE, CRI, MLE and bootstrap CI of λ_2 are not at all satisfactory for small values of n and small $\tau_2 - \tau_1$. However, BE and CRI work quite well for moderate or large sample sizes and for large $\tau_2 - \tau_1$ and the performance of BE and CRI is close to that of MLE and bootstrap CI of λ_2 in these cases. It is also noticed that HPD CRI works well for λ_1, where symmetric CRI works well for λ_2. Therefore we recommend to use HPD CRI for λ_1 and symmetric CRI for λ_2. Note that one may generate α form other distributions having support on (0,1) instead of uniform distribution as mentioned in Algorithm 4.4.1. A right truncated gamma distribution and the distribution obtained by spline fitting to the posterior distribution of α have been tried. However, no significant improvement has been noticed.

Chapter 5

Simple Step-stress Model for Two-Parameter Exponential Distribution[1]

5.1 Introduction

In the previous chapter we have considered the analysis of a simple step-stress model, when the lifetimes of the experimental units follow one-parameter exponential distribution. The purpose of this chapter is to consider the analysis of a simple step-stress model based on the assumption that the lifetime of the experimental units follows two-parameter exponential distribution. The analysis has been performed based on the assumption that the model satisfies CEM assumption, and the data are Type-II censored. One of the justifications for incorporating the location parameter is the presence of possible bias in the experimental data due to calibration. It is assumed that as the stress level changes from s_1 to s_2, the scale parameter of the exponential distribution changes from θ_1 to θ_2, but the location parameter μ remains unchanged. The data are assumed to be Type-II censored. It is observed that the MLEs of the unknown parameters do not always exist. Whenever they exist, they can be obtained in closed form. We obtain the exact conditional distributions of the MLEs of

[1] Part of this work is under revision in *Statistical Methodology*.

the scale parameters. Since the conditional distributions of the MLEs of the scale parameters depend on the unknown location parameter μ, it is not possible to obtain the exact CIs of the scale parameters based on the exact conditional distributions. We propose to use the Fisher information matrix to construct the asymptotic CIs of the unknown scale parameters, assuming the location parameter to be known. We also propose to use the parametric bootstrap method for constructing CI for the scale parameters, and it is very easy to implement in practice. We further consider the Bayesian inference of the unknown parameters θ_1, θ_2 and μ. It is assumed that θ_2 has an inverted gamma prior, and α has a beta prior, where $\theta_1\alpha = \theta_2$. The location parameter μ is assumed to have a non-informative prior. Based on the above priors the Bayes estimates and the associate credible intervals are obtained using importance sampling technique. Extensive simulations are performed to compare the performances of the different methods and the performances are quite satisfactory. One data analysis has been performed for illustrative purposes.

Rest of the chapter is organized as follows. In Section 5.2 first we discuss the model formulation and then provide the MLEs of the three unknown parameters. The conditional distribution of the MLEs of the scale parameters are presented in Section 5.3. In Section 5.4 we discuss the construction of different confidence intervals for the scale parameters. Bayesian inference of the model parameters is indicated in Section 5.5. Simulation results and a data analysis are provided in Section 5.6, and finally conclusions appear in Section 5.7.

5.2 Model Description and MLEs

5.2.1 Model Description

We consider a simple SSLT, where n identical units are placed on a life testing experiment at the initial stress level s_1. The stress level is changed to a higher level s_2 at a prefixed time τ. Let $r(\leq n)$ be a prefixed positive integer. Further, the experiment is terminated as soon as the rth failure occurs. The failure times $t_{1:n} < \cdots < t_{r:n}$ denote the observed data. The following cases may be observed:

(a) $t_{1:n} < \cdots < t_{r:n} < \tau$,

(b) $t_{1:n} < \cdots < t_{N:n} < \tau < t_{N+1:n} < \cdots < t_{r:n}$,

(c) $\tau < t_{1:n} < \cdots < t_{r:n}$,

where N is the number of failures at the stress level s_1. Note that for Case (a) and Case (c), $N = r$ and $N = 0$, respectively.

We also assume that the lifetime distributions at the stress levels s_1 and s_2 are exponential with scale parameters θ_1 and θ_2, respectively and a common location parameter μ. The lifetimes of the experimental units are assumed to be independently distributed. Then under the assumption of the CEM, the CDF, $F_T(\cdot)$, of a lifetime of an item is given by

$$F_T(t) = \begin{cases} 1 - e^{-\frac{t-\mu}{\theta_1}} & \text{if } \mu < t \leq \tau \\ 1 - e^{-\frac{t-\tau}{\theta_2} - \frac{\tau-\mu}{\theta_1}} & \text{if } \tau < t < \infty, \end{cases} \tag{5.1}$$

when $\mu < \tau$. When $\mu \geq \tau$ the same is given by

$$F_T(t) = 1 - e^{-\frac{t-\mu}{\theta_2}} \quad \text{if } t > \mu.$$

The corresponding PDF, $f_T(t)$, is given by

$$f_T(t) = \begin{cases} \frac{1}{\theta_1} e^{-\frac{t-\mu}{\theta_1}} & \text{if } \mu < t \leq \tau \\ \frac{1}{\theta_2} e^{-\frac{t-\tau}{\theta_2} - \frac{\tau-\mu}{\theta_1}} & \text{if } \tau < t < \infty. \end{cases} \tag{5.2}$$

when $\mu < \tau$, when $\mu \geq \tau$ the same is given by

$$f_T(t) = \frac{1}{\theta_2} e^{-\frac{t-\mu}{\theta_2}} \quad \text{if } t > \mu.$$

5.2.2 Likelihood Function and MLEs

In this section we consider the likelihood function of the observed data and obtain the MLEs of the unknown parameters. Note that if $\mu > \tau$, $\theta_1 > 0$ and $\theta_2 > 0$ the likelihood of the observed data is given by

$$L(\mu,\, \theta_1,\, \theta_2) = \frac{1}{\theta_2} e^{-\frac{1}{\theta_2}\left\{\sum_{j=1}^{r} t_{i:n} + (n-r)t_{r:n} - n\mu\right\}},$$

which is maximum at $\mu = t_{1:n}$, $\theta_2 = \frac{1}{r}\left\{\sum_{j=1}^{r} t_{j:n} + (n-r)t_{r:n} - nt_{1:n}\right\}$ for any value of θ_1, as the likelihood does not depends on θ_1. This is actually equivalent to a CS-II life test under the second stress level. For this reason we assume that $\mu < \tau$, so that the experiment is a proper simple SSLT. Now using (5.1) and (5.2), the likelihood of the observed data is given by

$$L(\mu, \theta_1, \theta_2) = \begin{cases} \dfrac{1}{\theta_2^r}\, e^{-\frac{n}{\theta_1}\tau + \frac{n}{\theta_1}\mu - \frac{D_2}{\theta_2}} & \text{if } N = 0 \\ \dfrac{1}{\theta_1^N \theta_2^{r-N}} e^{-\frac{D_1}{\theta_1} - \frac{n}{\theta_1}(t_{1:n}-\mu) - \frac{D_2}{\theta_2}} & \text{if } 1 \le N \le r-1 \\ \dfrac{1}{\theta_1^r}\, e^{-\frac{D_1}{\theta_1} - \frac{n}{\theta_1}(t_{1:n}-\mu)} & \text{if } N = r, \end{cases} \tag{5.3}$$

where $D_1 = \displaystyle\sum_{j=1}^{N} t_{j:n} + (n-N)m - nt_{1:n}$, $D_2 = \displaystyle\sum_{j=N+1}^{r} t_{j:n} + (n-r)t_{r:n} - (n-N)m$, and $m = \min\{\tau,\, t_{r:n}\}$. For $N = 0$ and for fixed θ_1 and θ_2, $L(\mu,\, \theta_1,\, \theta_2)$ is maximum at $\mu = t_{1:n} > \tau$. Now for $N = 0$

$$L(t_{1:n},\, \theta_1,\, \theta_2) = \frac{1}{\theta_2^r} e^{\frac{1}{\theta_1} n(t_{1:n}-\tau) - \frac{D_2}{\theta_2}} \qquad \theta_1 > 0,\, \theta_2 > 0,$$

which increases as θ_1 decreases. Hence, there exists a path along which $L(\mu,\, \theta_1,\, \theta_2)$ in (5.3) increases for $N = 0$, and MLE of $(\mu,\, \theta_1,\, \theta_2)$ does not exist in this case. Similarly, MLE of $(\mu,\, \theta_1,\, \theta_2)$ does not exist also for $N = r$. For $1 \le N \le r-1$, MLE of $(\mu,\, \theta_1,\, \theta_2)$ exists and is given by $(\widehat{\mu},\, \widehat{\theta}_1,\, \widehat{\theta}_2)$, where

$$\widehat{\mu} = t_{1:n}, \qquad \widehat{\theta}_1 = \frac{D_1}{N}, \qquad \text{and} \qquad \widehat{\theta}_2 = \frac{D_2}{r-N}. \tag{5.4}$$

Clearly this MLE is conditional MLE of $(\mu, \theta_1, \theta_2)$ conditioning on the event $1 \le N \le r-1$.

5.3 Conditional Distribution of MLEs

In this section we provide the marginal distributions of the MLEs conditioning on $1 \le N \le r-1$. It can be obtained by inverting the conditional MGFs as it was first suggested by Bartholmew [39]. Note that conditional distribution of $\widehat{\mu} = t_{1:n}$ is same as that of lowest order statistics of a sample of size n from two-parameter exponential distribution with location parameter μ and scale parameter θ_1, conditioning on the event that it lies between μ and τ. As this distribution has been well studied in literature, we do not pursue it under frequentist setup in this dissertation. The conditional MGF of $\widehat{\theta}_1$, conditioning on the event $A = \{1 \le N \le r-1\}$, can be written as

$$E[e^{\omega\widehat{\theta}_1}|A] \;=\; \sum_{i=1}^{r-1} E[e^{\omega\widehat{\theta}_1}|N=i] \times P[N=i|1 \le N \le r-1]. \qquad (5.5)$$

Now the number of the failures before time τ, N, is a non-negative random variable with PMF

$$P[N=i] = \binom{n}{i} (1-e^{-\frac{\tau-\mu}{\theta_1}})^i \, e^{-(n-i)\frac{\tau-\mu}{\theta_1}} = p_i \,(\text{say}) \quad \text{for} \quad i = 0, 1, \cdots, n,$$

so that for $i = 1, \ldots, r-1$

$$P[N=i|1 \le N \le r-1] = \frac{p_i}{\sum_{j=1}^{r-1} p_j}.$$

The exact derivations of $E[e^{\omega\widehat{\theta}_1}|A]$ is provided in Appendix 5.A. Using the inversion formula, the exact conditional distribution of $\widehat{\theta}_1$ can be obtained from conditional MGF and the corresponding PDF is given in Theorem 5.3.1.

Theorem 5.3.1. The PDF of $\widehat{\theta}_1$ conditioning on $A = \{1 \leq N \leq r-1\}$ is given by

$$\begin{aligned} f_{\widehat{\theta}_1}(t) &= c_{10}\, f_4\left(t - \tau_{10}; \theta_1\,(n-1)\right) - d_{10}\, f_4\left(t; \theta_1\,(n-1)\right) \\ &\quad + \sum_{i=2}^{r-1}\sum_{j=0}^{i-1} c_{ij}\, f_3\left(t - \tau_{ij}; i-1, \frac{\theta_1}{i}, \frac{(n-j-1)\theta_1}{i(j+1)}\right) \\ &\quad - \sum_{i=2}^{r-1}\sum_{j=0}^{i-1} d_{ij}\, f_3\left(t; i-1, \frac{\theta_1}{i}, \frac{(n-j-1)\theta_1}{i(j+1)}\right), \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} d_{ij} &= \tfrac{(-1)^{i-j-1}}{\sum_{k=1}^{r-1} p_k}\binom{n}{i}\binom{i}{j+1}\, e^{-\frac{n}{\theta_1}(\tau-\mu)}, \\ c_{ij} &= \tfrac{(-1)^{i-j-1}}{\sum_{k=1}^{r-1} p_k}\binom{n}{i}\binom{i}{j+1}\, e^{-\frac{n-j-1}{\theta_1}(\tau-\mu)}, \\ \tau_{ij} &= \tfrac{1}{i}(n-j-1)(\tau-\mu), \end{aligned} \tag{5.7}$$

$$f_3(t;\eta,\xi_1,\xi_2) = \frac{1}{\xi_2\,\left(1+\xi_1/\xi_2\right)^{\eta}}\, e^{t/\xi_2} \int_{\max\{0,(1/\xi_1+1/\xi_2)t\}}^{\infty} \frac{1}{\Gamma(\eta)} z^{\eta-1} e^{-z} \mathrm{d}z \quad \text{for } t \in \mathbb{R},$$

and

$$f_4(t;\xi) = \begin{cases} \frac{1}{\xi} e^{t/\xi} & \text{if t} \in (-\infty, 0) \\ 0 & \text{otherwise.} \end{cases}$$

∎

Proof: See Appendix 5.A.

Similarly, inverting the conditional MGF of $\widehat{\theta}_2$, conditional PDF of $\widehat{\theta}_2$ is given in Theorem 5.3.2.

Theorem 5.3.2. The PDF of $\widehat{\theta}_2$ conditioning on $\{1 \leq N \leq r-1\}$ is given by

$$f_{\widehat{\theta}_2}(t) = \sum_{i=1}^{r-1} c_i\, f_1\left(t, r-i, \frac{\theta_2}{r-i}\right),$$

where $c_i = \dfrac{p_i}{\sum_{k=1}^{r-1} p_k}$ and $f_1(t,\eta,\xi) = \dfrac{1}{\xi^{\eta}\,\Gamma(\xi)}\, t^{\eta-1}\, e^{-t/\xi}$ if $t > 0$. ∎

Proof: See Appendix 5.A.

Since the shape of the conditional PDF of $\widehat{\theta}_1$ as given in Theorem 5.3.1 is difficult to analyze analytically, we provide the plots in Figure 5.1 of the PDFs of $\widehat{\theta}_1$ for $n = 20$, $\mu = 0$, $\theta_1 = 12$, $\theta_2 = 4.5$, and $r = 20$ (complete sample). We consider four different values of τ, *viz.*, 4, 6, 8, and 10. For comparison purposes, we have also generated samples from the same CEM model as given in (5.1), and compute the MLEs of θ_1, θ_2 and μ, whenever they exist. We provide the histograms of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ based on 10000 replications along with the true conditional PDFs of $\widehat{\theta}_1$ and $\widehat{\theta}_2$. It is clear that the true PDFs match very well with the corresponding histograms. The PDF plot of $\widehat{\theta}_2$ which is a mixture of gamma distributions is provided in Figure 5.2 for the above mentioned parameters.

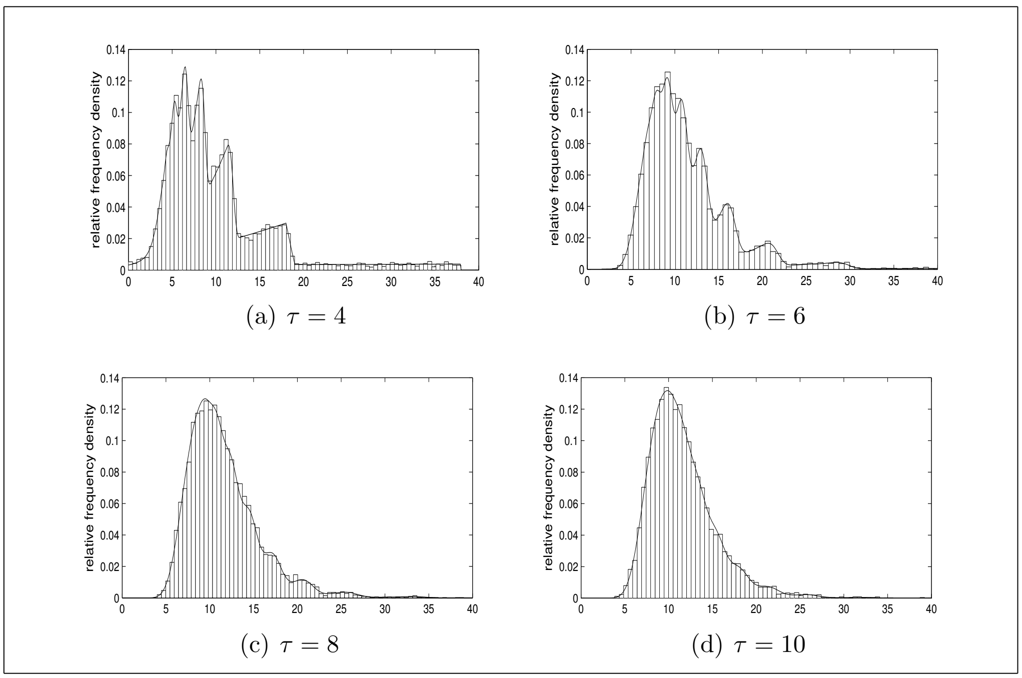
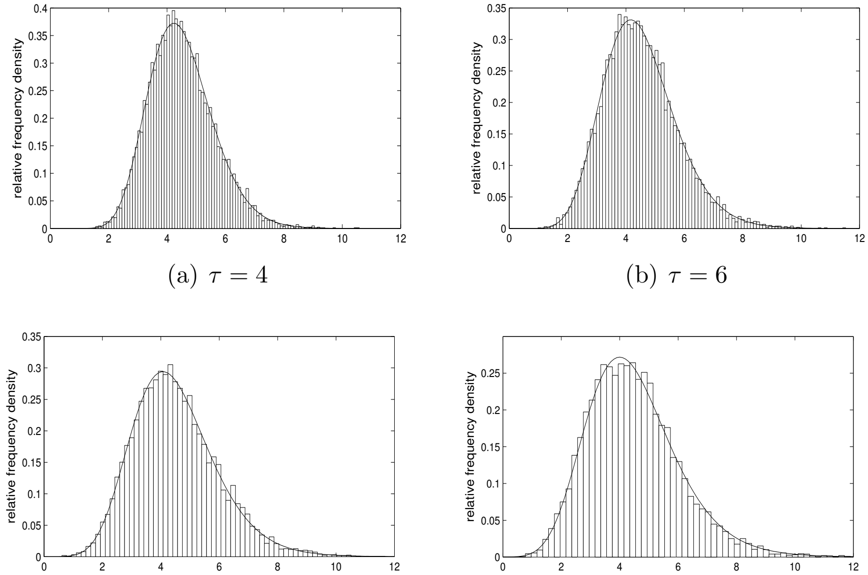


Figure 5.1: PDF-plot of $\widehat{\theta}_1$ for different values of τ and for $n = r = 20$, $\mu = 0$, $\theta_1 = 12$, and $\theta_2 = 4.5$.



(a) $\tau = 4$ (b) $\tau = 6$

(c) $\tau = 8$ (d) $\tau = 10$

Figure 5.2: PDF-plot of $\widehat{\theta}_2$ for different values of τ and for $n = 20$, $r = 20$, $\mu = 0$, $\theta_1 = 12$, and $\theta_2 = 4.5$.

5.4 Different Types of Confidence Intervals

5.4.1 Asymptotic Confidence Interval

In the absence of a closed form of the conditional CDFs of the parameter estimates $\widehat{\theta}_1$ and $\widehat{\theta}_2$, we cannot obtain the exact CIs. Because of the complicated nature of these integrals, we cannot consider the tail probabilities of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ for the construction of exact CIs as in Chen and Bhattacharya [47]. Moreover, it is observed empirically that $P_{\theta_1}(\widehat{\theta}_1 < b)$ is not a monotone function of θ_1, which is depicted in the Figure 5.3 taking $\theta_1 = e^{-2.5}$, $\theta_2 = e^{-1.5}$, $\mu = 0$, $n = 20$, $r = 15$, $\tau = 4$, and $b = 5$. Hence, the construction of the exact confidence intervals become very difficult. Due to this reason, we proceed to obtain the asymptotic CIs of θ_1 and θ_2. We provide the elements of the Fisher information matrix. Though we have three parameters μ, θ_1, θ_2, we obtain the Fisher information matrix for θ_1 and θ_2 only, assuming μ is known and use the estimate of μ in the final expressions. We then use the asymptotic normality of the MLEs to construct asymptotic CIs of θ_1 and θ_2. For the purpose of comparison we also use parametric bootstrap methods, see Efron and Tibshirani [68], to construct CIs for the two scale parameters.

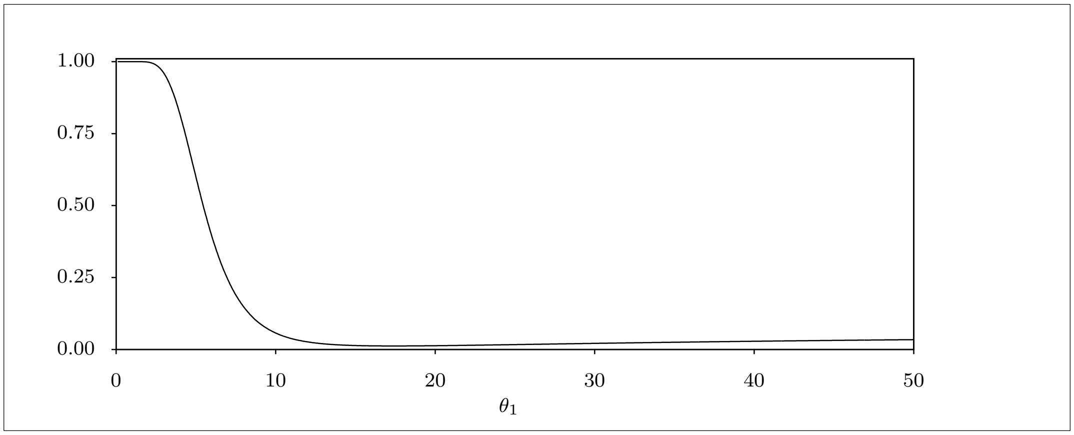


Figure 5.3: Non-monotonicity of tail probability of $\widehat{\theta}_1$ as a function of θ_1.

Let $I(\theta_1,\, \theta_2) = (I_{ij}(\theta_1,\, \theta_2))$; $i, j = 1, 2$ denote the Fisher information matrix of θ_1 and θ_2, where

$$I_{11}(\theta_1, \theta_2) = E\big[-\tfrac{N}{\theta_1^2} + \tfrac{2D_1}{\theta_1^3}\big], \qquad I_{12}(\theta_1, \theta_2) = 0,$$

$$I_{21}(\theta_1, \theta_2) = 0, \qquad I_{22}(\theta_1, \theta_2) = E\big[-\tfrac{r-N}{\theta_2^2} + \tfrac{2D_2}{\theta_2^3}\big].$$

The observed information matrix is

$$\begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} = \begin{bmatrix} \frac{N}{\widehat{\theta}_1^2} & 0 \\ 0 & \frac{r-N}{\widehat{\theta}_2^2} \end{bmatrix}.$$

An estimate of variance of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ can be obtained through the observed information matrix as

$$V_1 = \frac{\widehat{\theta}_1^2}{N} \quad \text{and} \quad V_2 = \frac{\widehat{\theta}_2^2}{r-N}.$$

The asymptotic distributions of the pivotal quantities $\frac{\widehat{\theta}_1 - E(\widehat{\theta}_1)}{\sqrt{V1}}$ and $\frac{\widehat{\theta}_2 - E(\widehat{\theta}_2)}{\sqrt{V2}}$ may then be used to construct $100(1-\gamma)\%$, $0 < \gamma < 1$, CIs for θ_1 and θ_2, respectively. The

$100(1-\gamma)\%$ confidence interval for θ_1 and θ_2 are given by

$$\left[\widehat{\theta}_1 \pm z_{1-\frac{\gamma}{2}}\sqrt{V_1}\right] \quad \text{and} \quad \left[\widehat{\theta}_2 \pm z_{1-\frac{\gamma}{2}}\sqrt{V_2}\right].$$

5.4.2 Bootstrap Confidence Interval

In this subsection, we construct bootstrap CIs based on parametric bootstrapping method. Later we show that bootstrap CIs has better coverage probabilities than asymptotic CIs unless the sample size is quite large. Now we describe the algorithm to obtain bootstrap CIs for θ_1 and θ_2.

Parametric Bootstrap:

Step 1. Given τ, n and the original sample $t = (t_{1:n}, t_{2:n}, \cdots, t_{r:n})$ obtain $\widehat{\mu}$, $\widehat{\theta}_1$ and $\widehat{\theta}_2$, the MLEs of μ, θ_1, and θ_2, respectively.

Step 2. Simulate a sample of size n from uniform $(0,1)$ distribution, denote the ordered sample as $U_{1:n}, U_{2:n}, \cdots, U_{n:n}$.

Step 3. Find N, such that $U_{N:n} \leq 1 - e^{-\frac{\tau-\widehat{\mu}}{\widehat{\theta}_1}} \leq U_{N+1:n}$.

Step 4. If $1 \leq N \leq r-1$, proceed to the next step. Otherwise go back to Step 2.

Step 5. For $j = 1, 2, \cdots, N$, $T_{j:n} = \widehat{\mu} - \widehat{\theta}_1 \ln(1-U_{j:n})$. For $j = N+1, \cdots, r$, $T_{j:n} = \tau - \frac{\widehat{\theta}_2}{\widehat{\theta}_1}(\tau - \widehat{\mu}) - \widehat{\theta}_2 \ln(1-U_{j:n})$.

Step 6. Compute the MLEs of θ_1 and θ_2 based on $T_{1:n}, T_{2:n}, \cdots, T_{r:n}$, say $\widehat{\theta}_1^{(1)}$ and $\widehat{\theta}_2^{(1)}$.

Step 7. Repeat Steps 2-5 M times and obtain $\widehat{\theta}_1^{(1)}, \widehat{\theta}_2^{(1)}, \widehat{\theta}_1^{(2)}, \widehat{\theta}_2^{(2)}, \cdots, \widehat{\theta}_1^{(M)}, \widehat{\theta}_2^{(M)}$.

Step 8. Arrange $\widehat{\theta}_1^{(1)}$, $\widehat{\theta}_1^{(2)}, \cdots, \widehat{\theta}_1^{(M)}$ in ascending order to obtain $\widehat{\theta}_1^{[1]}$, $\widehat{\theta}_1^{[2]}$, $\cdots$, $\widehat{\theta}_1^{[M]}$. Similarly, arrange $\widehat{\theta}_2^{(1)}$, $\widehat{\theta}_2^{(2)}$, $\cdots$, $\widehat{\theta}_2^{(M)}$ in ascending order to obtain

$\widehat{\theta}_2^{[1]}, \widehat{\theta}_2^{[2]}, \cdots, \widehat{\theta}_2^{[M]}$. A two-sided $100(1-\gamma)\%$ bootstrap confidence interval of θ_i, $(i = 1, 2)$ is then given by

$$\left(\widehat{\theta}_i^{[\frac{\gamma}{2}M]}, \widehat{\theta}_i^{[(1-\frac{\gamma}{2})M]}\right),$$

where $[x]$ denotes the largest integer less than or equal to x.

5.5 Bayesian Inference

As the conditional distribution of the MLEs of the unknown parameters are quite complicated, Bayesian analysis seems to be a reasonable alternative. Also it is well known that the bootstrap CI of the threshold parameter μ does not work well, but a proper Bayesian CRI for μ can be obtained in a standard manner. In this section we mainly consider the square error loss function, although any other loss functions can be considered in a similar fashion. Here we assume that the data are coming form the distribution as mentioned in (5.1). To proceed further, we need to make some prior assumptions on the unknown parameters. Note that the basic aim of the step-stress life tests is to get more failures at the higher stress level, hence, it is reasonable to assume that $\theta_1 > \theta_2$. One of the prior assumption that supports $\theta_1 > \theta_2$ is $\theta_1 = \dfrac{\theta_2}{\alpha}$, where $0 < \alpha < 1$. Here we assume that θ_2 has an inverted gamma (IG) distribution with parameters $a > 0$, $b > 0$, α has a beta distribution with parameters $c > 0$, $d > 0$, and the location parameter μ has a non-informative prior over $(-\infty, \tau)$. The prior density for θ_2, α and μ are given by

$$\begin{aligned}
\pi_1(\theta_2) &= \frac{b^a}{\Gamma(a)} \frac{e^{-b/\theta_2}}{\theta_2^{a+1}} && \text{if } \theta_2 > 0, \\
\pi_2(\alpha) &= \frac{1}{B(c,d)} \alpha^{c-1}(1-\alpha)^{d-1} && \text{if } 0 < \alpha < 1, \\
\pi_3(\mu) &= 1 && \text{if } -\infty < \mu < \tau,
\end{aligned}$$

respectively. We also assume that μ, α, and θ_2 are independently distributed. Likelihood function of the data for given $(\mu,\, \alpha,\, \theta_2)$ can be expressed as

$$l(\text{Data}|\mu,\, \alpha,\, \theta_2) \propto \begin{cases} \dfrac{\alpha^N}{\theta_2^r}\, e^{-\frac{1}{\theta_2}\{\alpha\, D_3 + D_2 - n\,\alpha\,\mu\}} & \text{if } \mu < t_{1:n} < \cdots < t_{N:n} < \tau \\ & \quad < t_{N+1:n} < \cdots < t_{r:n}, \\ & \quad 1 \le N \le r-1 \\ \dfrac{\alpha^r}{\theta_2^r}\, e^{-\frac{1}{\theta_2}\{\alpha\, D_3 - n\,\alpha\,\mu\}} & \text{if } \mu < t_{1:n} < \cdots < t_{r:n} < \tau \\ \dfrac{1}{\theta_2^r}\, e^{-\frac{1}{\theta_2}\{n\,\tau\,\alpha + D_2 - n\,\alpha\,\mu\}} & \text{if } \mu < \tau < t_{1:n} < \cdots < t_{r:n}, \end{cases}$$

where $D_3 = D_1 + nt_{1:n}$. Hence, for $0 < \alpha < 1, \theta_2 > 0$, the posterior density of the parameters given data can be written as

Case-I : $N = 0$

$$l(\mu,\, \alpha,\, \theta_2|\text{Data}) \;\; \propto \;\; \frac{1}{\theta_2^{r+a+1}}\, \alpha^{c-1}\, (1-\alpha)^{d-1}\, e^{-\frac{1}{\theta_2}\{n\,\alpha\,\tau + D_2 + b - n\,\alpha\,\mu\}} \text{ if } \mu < \tau.$$

Case-II : $N = 1,\, 2,\, \cdots,\, r$

$$l(\mu,\, \alpha,\, \theta_2|\text{Data}) \;\; \propto \;\; \frac{1}{\theta_2^{r+a+1}}\, \alpha^{N+c-1}\, (1-\alpha)^{d-1}\, e^{-\frac{1}{\theta_2}\{\alpha\, D_3 + D_2 + b - n\,\alpha\,\mu\}} \text{ if } \mu < t_{1:n}.$$

Note that $l(\mu,\, \alpha,\, \theta_2|\text{Data})$ is integrable if $r + a - 1 > 0$ and $N + c - 1 > 0$. The Bayes estimate $\widehat{g}(\mu,\, \alpha,\, \theta_2)$ of some function, say $g(\mu,\, \alpha,\, \theta_2)$, with respect to the square error loss function is the posterior expectation of $g(\mu,\, \alpha,\, \theta_2)$, *i.e.*, it can be expressed as

$$\widehat{g}(\mu,\, \alpha,\, \theta_2) \;\; = \;\; \int_0^1\int_0^\infty\int_{-\infty}^{t_{1:n}} g(\mu,\, \alpha,\, \theta_2)\, l(\mu,\, \alpha,\, \theta_2|\text{Data})d\mu\, d\theta_2\, d\alpha. \tag{5.8}$$

Unfortunately, (5.8) cannot be found explicitly for general function $g(\mu,\, \alpha,\, \theta_2)$. One can use numerical methods to compute (5.8). Alternatively, Lindley's approximation, see Lindley [99], can be used to approximate (5.8). However CRI cannot be found by any of the above methods. Hence, we propose importance sampling to compute the Bayes estimate and as well as to construct CRI. In Case-II, (5.8) can be written as

$$\widehat{g}(\mu,\, \alpha,\, \theta_2) \;=\; \frac{\displaystyle\int_0^1\int_0^\infty\int_{-\infty}^{t_{1:n}} g_1(\mu,\, \alpha,\, \theta_2) l_1(\alpha) l_2(\theta_2|\alpha) l_3(\mu|\alpha,\, \theta_2) d\mu\, d\theta_2\, d\alpha}{\displaystyle\int_0^1\int_0^\infty\int_{-\infty}^{t_{1:n}} g_2(\mu,\, \alpha,\, \theta_2) l_1(\alpha) l_2(\theta_2|\alpha) l_3(\mu|\alpha,\, \theta_2) d\mu\, d\theta_2\, d\alpha}, \quad (5.9)$$

where

$$\begin{aligned}
g_1(\mu,\, \alpha,\, \theta_2) &= \frac{g(\mu,\, \alpha,\, \theta_2)\, \alpha^{N+c-2}(1-\alpha)^{d-1}}{(D_1\alpha + D_2 + b)^{r+a-1}}, \\
g_2(\mu,\, \alpha,\, \theta_2) &= \frac{\alpha^{N+c-2}(1-\alpha)^{d-1}}{(D_1\alpha + D_2 + b)^{r+a-1}}, \\
l_1(\alpha) &= 1, \qquad 0 < \alpha < 1, \\
l_2(\theta_2|\alpha) &= \frac{(D_1\alpha + D_2 + b)^{r+a-1}}{\Gamma(r+a-1)} \times \frac{e^{-(D_1\alpha + D_2 + b)/\theta_2}}{\theta_2^{r+a}}, \qquad \theta_2 > 0, \\
l_3(\mu|\alpha,\, \theta_2) &= \frac{n\,\alpha}{\theta_2}\, e^{n\,\alpha(\mu - t_{1:n})/\theta_2}, \qquad \mu < t_{1:n}.
\end{aligned}$$

Note that $l_3(\mu|\alpha,\, \theta_2)$ has a closed and invertible distribution function, and hence, one can easily draw sample from this density function. It may be noted that the above choice of g_1, g_2, l_1, l_2, and l_3 functions are not unique, but the performance based on them are quite satisfactory.

Algorithm 5.5.1

Step 1. Generate α from U(0, 1).

Step 2. Generate θ_2 from IG$(r+a-1,\, D_1\alpha + D_2 + b)$.

Step 3. Generate μ from $l_3(\mu|\alpha,\, \theta_2)$.

Step 4. Repeat steps 1-3 M times to obtain $\{(\mu_1,\, \alpha_1,\, \theta_{21}),\, \ldots,\, (\mu_M,\, \alpha_M,\, \theta_{2M})\}$.

Step 5. Calculate $g_i = g_1(\mu_i,\, \alpha_i,\, \theta_{2i})$ for $i = 1,\, 2,\, \ldots,\, M$.

Step 6. Calculate $w_i = g_2(\mu_i,\, \alpha_i,\, \theta_{2i})$ for $i = 1,\, 2,\, \ldots,\, M$.

Step 7. Calculate normalizing weight $w_i^* = \dfrac{w_i}{\sum_{j=1}^M w_j}$ for $i = 1,\, 2,\, \ldots,\, M$.

Step 8. Approximate (5.8) by

$$\widehat{g}(\mu,\, \alpha,\, \theta_2) = \sum_{i=i}^{M} w_i^* g_i.$$

Step 9. To find a $100\,(1-\gamma)\%$, $0 < \gamma < 1$, CRI for $g(\mu,\, \alpha,\, \theta_2)$, arrange the $\{g_1,\, g_2,\, \ldots,\, g_M\}$ to get $\left\{g_{(1)} < g_{(2)} < \ldots < g_{(M)}\right\}$. Arrange $\{w_1^*,\, w_2^*,\, \ldots,\, w_M^*\}$ accordingly to get $\left\{w_{(1)}^*,\, w_{(2)}^*,\, \ldots,\, w_{(M)}^*\right\}$. Note that $w_{(i)}^*$'s are not ordered. A $100\,(1-\gamma)\%$ CRI is then given by $\left(g_{(j_1)},\, g_{(j_2)}\right)$, where $1 \le j_1 \le M$ and $1 \le j_2 \le M$ satisfy

$$j_1 < j_2, \qquad \sum_{i=j_1}^{j_2} w_{1(i)} \le 1-\gamma < \sum_{i=j_1}^{j_2+1} w_{1(i)}. \tag{5.10}$$

The $100(1-\gamma)\%$ HPD CRI of $g(\alpha,\, \lambda_2)$ becomes $(g_{(j_1^*)},\, g_{(j_2^*)})$, where $1 \le j_1^* < j_2^* \le M$ satisfy

$$\sum_{i=j_1^*}^{j_2^*} w_{1(i)} \le 1-\gamma < \sum_{i=j_1^*}^{j_2^*+1} w_{1(i)}, \qquad g_{(j_2^*)} - g_{(j_1^*)} \le g_{(j_2)} - g_{(j_1)},$$

for all j_1 and j_2 satisfying (5.10).

For Case-I, (5.8) can be expressed in the same fashion as given in (5.9) with

$$\begin{aligned}
g_1(\mu,\, \alpha,\, \theta_2) &= \frac{1}{\alpha}\, g(\mu,\, \alpha,\, \theta_2),\\
g_2(\mu,\, \alpha,\, \theta_2) &= \frac{1}{\alpha},\\
l_1(\alpha) &= \frac{1}{B(c,\, d)} \alpha^{c-1}(1-\alpha)^{d-1}, \qquad 0 < \alpha < 1,\\
l_2(\theta_2|\alpha) &= \frac{(D_2+b)^{r+a-1}}{\Gamma(r+a-1)}\, \frac{e^{-(D_2+b)/\theta_2}}{\theta_2^{r+a}}, \qquad \theta_2 > 0,\\
l_3(\mu|\alpha,\, \theta_2) &= \frac{n\,\alpha}{\theta_2}\, e^{-n\,\alpha(\tau-\mu)/\theta_2}, \qquad \mu < \tau.
\end{aligned}$$

Hence, Bayes estimate and credible interval for $g(\mu,\, \alpha,\, \theta_2)$ can be found using importance sampling in Case-I in the same manner as in Case-II.

5.6 Simulation Results and Data Analysis

To evaluate the performance of the CIs and CRIs we conduct simulation studies to obtain the CP and AL of the CIs and CRIs described in Section 5.4. The results are

Table 5.1: CP and AL of bootstrap and asymptotic confidence interval along with AE and MSE for MLE of θ_1.

					BCI				ACI			
					95%		99%		95%		99%	
n	r	τ	AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL
30	30	2.5	11.961	48.339	92.46	22.365	98.20	35.873	84.10	23.112	90.90	30.374
		3.0	11.962	38.754	92.72	21.315	98.38	35.269	84.42	20.647	91.48	27.135
		3.5	11.984	33.486	92.36	20.527	98.38	34.286	86.22	19.134	91.66	25.147
30	20	2.5	11.961	48.339	92.22	22.722	98.26	36.159	84.36	23.532	90.62	30.927
		3.0	11.962	38.754	93.22	21.977	98.34	36.415	86.18	21.007	92.08	27.607
		3.5	11.984	33.486	93.56	20.231	98.66	33.849	86.26	18.652	93.06	24.513
40	40	2.5	12.025	36.666	93.82	20.608	98.90	34.520	87.22	19.212	92.90	25.249
		3.0	12.010	27.018	94.02	19.224	98.74	32.005	87.54	17.418	93.52	22.892
		3.5	11.982	22.815	93.96	17.491	98.70	28.649	88.72	15.899	94.10	20.894
40	26	2.5	12.025	36.666	93.04	20.584	98.62	34.535	86.24	19.078	92.14	25.072
		3.0	12.010	27.018	93.52	19.110	98.92	32.063	87.80	17.292	93.04	22.726
		3.5	11.982	22.815	93.54	17.439	98.72	28.653	88.28	15.782	93.66	20.741
50	50	2.5	12.040	24.564	93.82	18.475	98.74	30.612	87.48	16.757	93.50	22.023
		3.0	11.993	18.326	94.10	16.599	98.88	26.894	89.28	15.084	94.02	19.824
		3.5	11.968	14.545	94.08	15.250	98.72	23.995	90.18	13.964	94.78	18.351
50	33	2.5	12.040	24.564	93.82	18.474	98.74	30.610	87.50	16.756	93.50	22.022
		3.0	11.993	18.326	93.62	16.589	98.84	26.954	88.30	15.039	93.66	19.764
		3.5	11.968	14.545	94.22	15.293	98.90	24.077	90.10	13.994	94.96	18.391

Table 5.2: CP and AL of bootstrap and asymptotic confidence interval along with AE and MSE for MLE of θ_2.

					BCI				ACI			
					95%		99%		95%		99%	
n	r	τ	AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL
30	30	2.5	4.486	0.811	94.30	3.609	98.34	4.776	93.10	3.591	96.98	4.719
		3.0	4.484	0.841	94.54	3.687	98.24	4.885	93.24	3.664	96.92	4.815
		3.5	4.482	0.883	94.24	3.753	98.02	4.977	93.00	3.726	96.52	4.896
30	20	2.5	4.502	1.374	93.28	4.740	97.56	6.355	91.50	4.661	95.50	6.126
		3.0	4.499	1.458	92.78	5.006	97.50	6.761	90.88	4.893	94.98	6.431
		3.5	4.498	1.621	93.46	5.253	97.50	7.152	91.06	5.091	95.08	6.691
40	40	2.5	4.483	0.609	94.40	3.110	98.38	4.108	93.62	3.101	97.40	4.075
		3.0	4.483	0.636	93.88	3.162	98.24	4.184	92.64	3.151	96.94	4.141
		3.5	4.482	0.660	94.42	3.257	98.48	4.309	93.46	3.241	97.40	4.259
40	26	2.5	4.492	1.064	93.72	4.182	98.12	5.578	92.10	4.132	96.24	5.430
		3.0	4.492	1.144	93.60	4.362	97.90	5.843	91.94	4.289	96.10	5.636
		3.5	4.490	1.243	93.28	4.583	97.46	6.181	91.22	4.476	95.62	5.882
50	50	2.5	4.482	0.496	94.68	2.779	98.52	3.667	94.12	2.772	97.84	3.643
		3.0	4.481	0.517	93.94	2.843	98.34	3.756	93.38	2.833	97.56	3.723
		3.5	4.480	0.537	94.04	2.903	98.32	3.837	93.34	2.891	97.46	3.799
50	33	2.5	4.487	0.839	93.92	3.686	98.14	4.894	92.84	3.654	96.76	4.802
		3.0	4.485	0.906	94.08	3.838	98.10	5.111	92.70	3.791	96.70	4.983
		3.5	4.485	0.979	93.98	3.982	98.26	5.326	92.42	3.920	96.40	5.152

based on 5000 simulations with $\mu = 0$, $\theta_1 = 12$, $\theta_2 = 4.5$, $M = 3000$. We consider different values of n, *viz.*, 30, 40, and 60 and different values for τ, *viz.*, 2.5, 3.0, and 3.5. For each value of n, we consider $r = n$ and $r = 0.65\,n$. We choose $a = b = 0$, $c = d = 1$, and $M = 8000$ for Bayesian analysis. Non-informative priors are chosen so that comparison with frequentist approach can be carried out. The results are provided in Tables 5.1, 5.2, 5.3, 5.4, and 5.5.

Table 5.3: CP and AL of different CRIs along with AE and MSE for BE of μ.

					Per. CRI				HPD CRI			
					95%		99%		95%		99%	
n	r	τ	AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL
30	30	2.5	-0.116	0.247	95.72	2.266	99.38	4.277	95.56	1.653	99.14	3.255
		3.0	-0.089	0.242	94.82	2.126	99.24	3.832	94.58	1.583	99.02	2.974
		3.5	-0.058	0.213	94.58	1.955	98.86	3.391	94.32	1.482	99.04	2.686
30	20	2.5	-0.148	0.271	96.30	2.399	99.52	4.545	96.12	1.751	99.60	3.454
		3.0	-0.116	0.248	95.18	2.197	98.92	3.954	95.66	1.638	99.18	3.077
		3.5	-0.090	0.231	95.44	2.075	99.06	3.578	95.44	1.573	99.20	2.845
40	40	2.5	-0.061	0.129	94.98	1.527	99.12	2.652	94.80	1.152	98.84	2.099
		3.0	-0.039	0.112	94.40	1.401	98.92	2.317	94.82	1.077	98.92	1.880
		3.5	-0.031	0.113	94.78	1.339	98.88	2.171	94.22	1.040	98.90	1.781
40	26	2.5	-0.062	0.124	95.16	1.540	98.98	2.665	94.86	1.167	98.92	2.112
		3.0	-0.053	0.120	94.62	1.459	99.02	2.424	94.86	1.120	98.88	1.963
		3.5	-0.044	0.110	94.82	1.387	99.04	2.240	94.60	1.078	98.92	1.840
50	50	2.5	-0.037	0.077	95.20	1.139	99.08	1.873	94.86	0.878	98.84	1.525
		3.0	-0.027	0.070	94.12	1.081	98.88	1.726	94.52	0.846	98.82	1.426
		3.5	-0.020	0.067	94.16	1.036	98.82	1.619	94.50	0.816	98.92	1.354
50	33	2.5	-0.037	0.072	95.12	1.126	99.04	1.848	94.66	0.869	98.84	1.506
		3.0	-0.024	0.065	95.38	1.056	99.12	1.682	94.66	0.826	99.14	1.394
		3.5	-0.019	0.066	94.80	1.030	99.08	1.615	94.42	0.810	99.06	1.344

From Tables 5.1 and 5.2, we see that the bootstrap CIs perform better than asymptotic CIs in terms of CP, though AL is larger in case of the bootstrap CIs compared to asymptotic CIs. For fixed n and r as the value of τ increases, performance of CIs for θ_1 improves on the account of availability of more data points and as expected, that of θ_2 deteriorates, but very marginally.

Performance of Bayesian CRIs are quite satisfactory (see Tables 5.3, 5.4, and 5.5). It is noticed that for fixed n and r as τ increases the performance of Bayes estimator and CRI of μ and θ_1 get improved, whereas performance of that of θ_2 get deteriorated, in the sense that MSE of the estimator and AL of corresponding

Table 5.4: CP and AL of different CRIs along with AE and MSE for BE of θ_1.

					Per. CRI				HPD CRI			
					95%		99%		95%		99%	
n	r	τ	AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL
30	30	2.5	15.079	105.064	96.40	35.701	99.38	66.879	93.30	28.138	98.66	52.901
		3.0	14.601	89.321	95.42	30.692	99.32	54.440	93.28	25.038	98.60	44.213
		3.5	13.857	67.058	95.52	25.760	99.38	43.465	92.98	21.663	98.96	36.356
30	20	2.5	15.954	122.545	97.40	38.004	99.60	72.048	95.54	30.000	99.18	56.524
		3.0	15.128	99.949	96.48	31.652	99.30	55.989	95.14	25.943	98.86	45.593
		3.5	14.656	71.814	96.28	27.416	99.34	45.724	95.72	23.120	98.94	38.491
40	40	2.5	14.272	77.026	95.14	27.478	99.14	46.506	92.72	22.927	98.62	38.835
		3.0	13.511	46.246	95.06	22.233	99.28	35.339	93.00	19.320	98.58	30.584
		3.5	13.174	41.242	94.60	19.632	98.98	30.246	92.88	17.378	98.58	26.653
40	26	2.5	14.492	72.320	96.22	27.394	99.38	45.984	95.06	23.003	99.08	38.598
		3.0	14.069	57.961	95.72	23.493	99.26	37.357	94.22	20.278	99.02	32.368
		3.5	13.657	38.565	95.68	20.319	99.18	31.043	94.66	18.034	98.92	27.539
50	50	2.5	13.847	48.502	94.98	22.162	99.30	34.815	94.10	19.360	98.80	30.308
		3.0	13.453	32.858	94.84	18.962	99.08	28.462	94.36	17.012	98.96	25.508
		3.5	13.078	22.031	94.86	16.590	98.98	24.224	93.66	15.148	98.88	22.133
50	33	2.5	13.683	45.501	95.32	21.787	99.14	34.319	94.10	19.048	98.80	29.879
		3.0	13.149	28.774	95.12	18.259	99.14	27.352	94.00	16.384	98.64	24.554
		3.5	12.962	24.986	94.96	16.407	99.02	24.059	93.70	14.969	98.82	21.899

Table 5.5: CP and AL of different CRIs along with AE and MSE for BE of θ_2.

					Per. CRI				HPD CRI			
					95%		99%		95%		99%	
n	r	τ	AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL
30	30	2.5	4.618	0.773	95.64	3.606	99.04	4.891	95.02	3.468	99.02	4.683
		3.0	4.647	0.831	95.42	3.708	99.22	5.031	94.90	3.565	98.94	4.816
		3.5	4.677	0.892	95.58	3.820	99.10	5.184	94.78	3.668	98.94	4.958
30	20	2.5	4.959	1.644	95.46	5.209	99.18	7.227	95.72	4.916	99.38	6.798
		3.0	5.040	1.840	95.28	5.458	99.14	7.571	96.22	5.142	99.20	7.117
		3.5	5.048	1.911	95.72	5.666	99.16	7.855	96.10	5.330	99.34	7.381
40	40	2.5	4.611	0.625	95.04	3.149	98.94	4.238	94.58	3.046	98.76	4.081
		3.0	4.630	0.661	94.92	3.238	99.04	4.361	94.52	3.128	98.92	4.197
		3.5	4.650	0.699	94.90	3.331	99.02	4.487	94.64	3.216	98.86	4.318
40	26	2.5	4.875	1.284	95.22	4.519	99.02	6.191	95.92	4.304	99.04	5.876
		3.0	4.913	1.383	95.60	4.727	99.10	6.474	96.08	4.494	99.50	6.144
		3.5	4.949	1.516	95.56	4.944	98.98	6.776	95.86	4.693	99.38	6.424
50	50	2.5	4.687	0.564	94.54	2.929	99.02	3.930	94.62	2.840	98.88	3.794
		3.0	4.701	0.598	94.48	3.008	98.98	4.035	94.66	2.913	98.96	3.894
		3.5	4.718	0.634	94.56	3.089	98.88	4.149	94.92	2.991	98.90	4.001
50	33	2.5	4.753	0.886	95.84	3.838	99.08	5.203	96.00	3.687	99.12	4.981
		3.0	4.783	0.970	95.92	4.014	99.14	5.444	95.84	3.850	99.06	5.209
		3.5	4.822	1.067	95.86	4.211	99.14	5.711	95.72	4.033	99.06	5.460

Table 5.6: Data for illustrative example.

Stress level	Data					
1	10.05	10.59	12.73	12.99	13.71	14.03
	14.34					
2	14.53	14.97	15.37	15.43	15.48	15.60
	15.76	16.18	16.46	16.86	16.90	17.02
	17.36	17.62	18.06	18.31	18.69	18.94
	18.95	22.65	22.89	24.51	25.39	

Table 5.7: Results of data analysis.

	Type	μ				θ_1				θ_2			
	of	95%		99%		95%		99%		95%		99%	
r	CI/CRI	LL	UL	LL	UL	LL	UL	LL	UL	LL	UL	LL	UL
30	ACI	–	–	–	–	4.46	29.98	0.45	33.96	2.07	4.93	1.62	5.38
	BCI	–	–	–	–	7.39	20.70	6.27	23.62	2.24	5.05	2.07	4.83
	Per.CRI	9.89	9.96	9.88	9.97	6.54	11.10	5.86	12.10	4.29	7.27	3.84	7.93
	HPD CRI	9.90	9.96	9.89	9.97	6.34	10.76	5.70	11.68	4.15	7.05	3.73	7.66
20	ACI	–	–	–	–	4.46	29.98	0.45	33.96	1.69	5.70	1.05	6.33
	BCI	–	–	–	–	7.39	20.70	6.27	23.62	2.09	5.27	1.90	5.90
	Per.CRI	9.84	9.94	9.83	9.95	8.15	14.78	7.30	15.52	5.34	9.69	4.78	10.17
	HPD CRI	9.84	9.93	9.83	9.94	8.28	14.84	7.59	15.55	5.43	9.73	4.97	10.19

CRI increase. Again for fixed τ, as n increases performance of estimator of all the parameters and all CRIs improve. As r increases the performance of all the estimators increases for fixed n and τ. We have also noticed that though the CP of the Bayesian CRIs are better than that of classical CIs, but the AL of classical CIs are less than that of Bayesian CRIs.

Next we provide a data analysis to illustrate the procedures described in sections 5.2, 5.4, and 5.5. A artificial data is generated from the CEM given in (5.1) with $n = 30$, $\mu = 10.0$, $\theta_1 = e^{2.5}$, $\theta_2 = e^{1.5}$, and $\tau = 14.5$ and is given in Table 5.6. We take $a = b = 0$, $c = d = 1$, and $M = 8000$. Based on the assumption that the data given in Table 5.6 is coming from exponential CEM, MLE of all the three parameters can be found using (5.4) and Bayes estimates can be found using the Algorithm 5.5.1. MLE of μ, θ_1 and θ_2 are 10.05, 17.22, and 3.50 and Bayes estimate of that are 9.40, 20.07, and 3.88, respectively. Asymptotic and bootstrap CI, symmetric and HPD CRI are also computed and reported in Table 5.7. In this case it is observed that bootstrap CIs and HPD CRIs are very similar for both the parameters θ_1 and θ_2.

5.7 Conclusion

The two-parameter exponential distribution has been considered in a simple step-stress model. Presence of the location parameter is justified in view of the possibility of an unknown bias in the lifetime experiment data. We obtain the exact distributions of the MLEs of the scale parameters at the two stress levels. The exact confidence limits of the scale parameters are difficult to obtain, due to the complicated nature of the model. We have proposed to use asymptotic and parametric bootstrap confidence intervals, and the performance of the later is better. We have further proposed Bayesian inference of the unknown parameters under fairly general prior assumptions, and we obtained the Bayes estimates and the associated credible intervals using importance sampling technique. The proposed Bayes estimates and the credible intervals perform quite well.

5.A Appendix

Lemma 5.A.1. Let $X_{1:n} < \cdots < X_{n:n}$ be the order statistics of a random sample of size n from a continuous distribution with PDF $f(x)$. Let D denote the number of order statistics less than or equal to some pre-fixed number τ, such that $F(\tau) > 0$, where $F(.)$ is the distribution function of $f(.)$. The conditional joint PDF of $X_{1:n}, \cdots, X_{D:n}$ given that $D = j$ is identical with the joint PDF of all order statistics of size j from the right truncated density function

$$f_*(t) = \begin{cases} \frac{f(t)}{F(\tau)} & \text{for } t < \tau \\ 0 & \text{otherwise.} \end{cases}$$

PROOF: See Balakrishnan et al. [27]

Lemma 5.A.2. Let $X_{1:n} < \cdots < X_{n:n}$ be the order statistics of a random sample of size n from a continuous distribution with PDF $f(x)$. Let D denote the number of order statistics less than or equal to some pre-fixed number τ, such that $F(\tau) > 0$, where $F(.)$ is the distribution function of $f(.)$. The conditional joint PDF of $X_{D+1:n}, \cdots, X_{n:n}$ given that $D = j$ is identical with the joint PDF of all order statistics of size $n - j$ from the left truncated density function

$$f_{**}(t) = \begin{cases} \frac{f(t)}{1-F(\tau)} & \text{for } \tau < t \\ 0 & \text{otherwise.} \end{cases}$$

PROOF: This can be proved following the same way of the prove of Lemma 5.A.1

Lemma 5.A.3. Let X be a Gamma(α, λ) random variable having the PDF

$$f_1(x;\, \alpha,\, \lambda) = \begin{cases} \frac{1}{\lambda^\alpha\, \Gamma(\alpha)}\, x^{\alpha-1}\, e^{-x/\lambda} & \text{for } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then for any arbitrary constant A, the MGF of $A + X$ is given by

$$M_{A+X} \;=\; e^{A\,\omega}\, (1 - \lambda\, \omega)^{-\alpha} \quad \text{for } \omega < \tfrac{1}{\lambda}$$

PROOF: It can be proved by simple integration and hence, the proof is omitted.

Lemma 5.A.4. Let X be an Exponential(λ) random variable having PDF

$$f_2(y;\, \lambda) = \begin{cases} e^{-\lambda y} & \text{for } y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then for any arbitrary constant A, the MGF of $A - X$ is given by

$$M_{A-X}\,(\omega) = e^{\omega A}\, (1 + \lambda\, \omega)^{-1} \quad \text{for} \quad \omega > -\frac{1}{\lambda}.$$

PROOF: It can be proved by simple integration and hence, the proof is omitted.

Corollary 5.A.1. Let X be a Gamma(α, λ_1) random variable, Y be an Exponential random variable with mean λ_2^{-1} and they are independently distributed. Then for any arbitrary constant A, the MGF of $A+X-Y$ is

$$M_{A+X-Y}(\omega) = e^{\omega A}\,(1-\lambda_1\,\omega)^{-\alpha}\,(1+\lambda_2\,\omega)^{-1} \quad \text{for } -\frac{1}{\lambda_2} < \omega < \frac{1}{\lambda_1}.$$

PROOF: Using Lemmas 5.A.3 and 5.A.4, it can be proved easily.

Lemma 5.A.5. Let X be a Gamma(α, λ_1), Y an Exponential(λ_2) random variable and they are independently distributed. Then the PDF of $X-Y$ is given by

$$f(t;\alpha,\lambda_1,\lambda_2) = \frac{1}{\lambda_1^{\alpha}\,\lambda_2\,\Gamma(\alpha)}\,e^{t/\lambda_2}\int_{\max\{0,t\}}^{\infty} z^{\alpha-1}e^{-(\frac{1}{\lambda_1}+\frac{1}{\lambda_2})z}dz \quad \text{for } t\in\mathbb{R}.$$

PROOF: It can be proved using transformation of variable.

Proof of Theorem 5.3.1

Using the lemma 5.A.1, we get

$$\begin{aligned}
E[e^{\omega\widehat{\theta_1}}|N=1] &= \frac{1}{\theta_1\left(1-e^{-\frac{\tau-\mu}{\theta_1}}\right)}\int_{\mu}^{\tau} e^{\omega(t_1-nt_1+(n-1)\tau)-\frac{1}{\theta_1}(t_1-\mu)}dt_1 \\
&= \frac{e^{-\frac{1}{\theta_1}(\tau-\mu)}}{\theta_1\left(1-e^{-\frac{\tau-\mu}{\theta_1}}\right)}\int_{\mu}^{\tau} e^{\left(\omega n-\omega+\frac{1}{\theta_1}\right)(t_1-\tau)}dt_1 \\
&= \frac{e^{-\frac{1}{\theta_1}(\tau-\mu)}}{\theta_1\left(1-e^{-\frac{\tau-\mu}{\theta_1}}\right)}\times\frac{e^{\left(\omega n-\omega+\frac{1}{\theta_1}\right)(\tau-\mu)}-1}{\omega n-\omega+\frac{1}{\theta_1}}. \qquad (5.11)
\end{aligned}$$

Using the lemma 5.A.1, we get for $i=2,\,3,\,\ldots,\,r-1$

$$E[e^{\omega\widehat{\theta_1}}|N=i] = \frac{i!}{\theta_1^i\left(1-e^{-\frac{\tau-\mu}{\theta_1}}\right)^i} \times\int_{\mu}^{\tau}\int_{t_1}^{\tau}\cdots\int_{t_{i-2}}^{\tau}\int_{t_{i-1}}^{\tau} e^{\frac{\omega}{i}(\sum_{j=1}^{i}t_j-nt_1+(n-i)\tau)-\frac{1}{\theta_1}\sum_{j=1}^{i}(t_j-\mu)}dt_i\cdots dt_1$$

$$
\begin{aligned}
&= \frac{i! e^{-\frac{i}{\theta_1}(\tau-\mu)}}{\theta_1^i \left(1 - e^{-\frac{\tau-\mu}{\theta_1}}\right)^i} \\
&\qquad \times \int_\mu^\tau \int_{t_1}^\tau \cdots \int_{t_{i-2}}^\tau \int_{t_{i-1}}^\tau e^{-\left(\frac{\omega n}{i} - \frac{\omega}{i} + \frac{1}{\theta_1}\right)(t_1-\tau) - \left(\frac{1}{\theta_1} - \frac{\omega}{i}\right)\sum_{j=2}^{i}(t_j-\tau)} dt_i \cdots dt_1 \\
&= \left(\frac{1}{\theta_1} - \frac{\omega}{i}\right)^{-1} \int_\mu^\tau \cdots \int_{t_{i-2}}^\tau e^{-\left(\frac{\omega n}{i} - \frac{\omega}{i} + \frac{1}{\theta_1}\right)(t_1-\mu) - \left(\frac{1}{\theta_1} - \frac{\omega}{i}\right)\sum_{j=2}^{i-1}(t_j-\tau)} \\
&\qquad\qquad\qquad \times \left\{ e^{-\left(\frac{1}{\theta_1} - \frac{\omega}{i}\right)(t_{i-1}-\tau)} - 1 \right\} dt_{i-1} \cdots dt_1 \\
&\vdots \\
&= \frac{e^{-\frac{i}{\theta_1}(\tau-\mu)}}{\left(1 - e^{-\frac{1}{\theta_1}(\tau-\mu)}\right)^i} \\
&\qquad \times \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{i}{j+1} \frac{e^{\left\{\frac{\omega}{i}(n-j-1) + \frac{1}{\theta_1}(j+1)\right\}(\tau-\mu)} - 1}{\left(1 - \frac{\omega\theta_1}{i}\right)^{i-1} \left(1 + \frac{\omega(n-j-1)\theta_1}{j(i+1)}\right)}.
\end{aligned}
\tag{5.12}
$$

Hence, using (5.5), (5.11), and (5.12), we have

$$
\begin{aligned}
E(e^{\omega\widehat{\theta}_1}|A) &= \sum_{i=1}^{r-1}\sum_{j=0}^{i-1} \frac{(-1)^{i-j-1}}{\sum_{k=1}^{n-1} p_k} \binom{n}{i}\binom{i}{j+1} e^{-\frac{n}{\theta_1}(\tau-\mu)} \frac{e^{\left\{\frac{\omega}{i}(n-j-1) + \frac{1}{\theta_1}(j+1)\right\}(\tau-\mu)} - 1}{\left(1 - \frac{\omega\theta_1}{i}\right)^{i-1}\left(1 + \frac{\omega(n-j-1)\theta_1}{i(j+1)}\right)} \\
&= \sum_{i=1}^{r-1}\sum_{j=0}^{i-1} c_{ij} \frac{e^{\omega\tau_{ij}}}{\left(1 - \frac{\theta_1\omega}{i}\right)^{i-1}\left(1 + \frac{(n-j-1)\theta_1\omega}{(j+1)i}\right)} \\
&\qquad - \sum_{i=1}^{r-1}\sum_{j=0}^{i-1} d_{ij} \frac{1}{\left(1 - \frac{\theta_1\omega}{i}\right)^{i-1}\left(1 + \frac{(n-j-1)\theta_1\omega}{(j+1)i}\right)}.
\end{aligned}
$$

where τ_{ij}, c_{ij} and d_{ij} are defined in (5.7). Now using Lemmas 5.A.3, 5.A.5 and Corollary 5.A.1, we have (5.6) and this completes the proof of the Theorem 5.3.1.

Proof of Theorem 5.3.2

CMGF of $\widehat{\theta}_2$ can be expressed as

$$
E[e^{\omega\widehat{\theta}_2}|A] = \sum_{i=1}^{r-1} E[e^{\omega\widehat{\theta}_2}|N = i] \times P(N = i|A).
$$

Using Lemma 5.A.2, for $i = 1, 2, \cdots, r-1$

$$\begin{aligned} E[e^{\omega\,\widehat{\theta_2}}|N=i] &= \frac{(n-i)!}{(n-r)!\,\theta_2^{r-i}}\, e^{-\left(\frac{\omega}{r-i}-\frac{1}{\theta_2}\right)(n-i)\tau} \\ &\quad \times \int_{\tau}^{\infty}\!\!\int_{t_{i+1}}^{\infty}\!\!\cdots\int_{t_{n-1}}^{\infty} e^{-\sum_{j=i+1}^{r-1}\left(\frac{1}{\theta_2}-\frac{\omega}{r-i}\right)t_{j:n}-\left(\frac{1}{\theta_2}-\frac{\omega}{r-i}\right)(n-r+1)t_{r:n}} dt_n\cdots dt_{i+1} \\ &= \frac{1}{\left(1-\frac{\theta_2\,\omega}{r-i}\right)^{r-i}}. \end{aligned}$$

Therefore

$$E[e^{\omega\widehat{\theta_2}}|A] = \sum_{i=1}^{r-1} \frac{1}{\left(1-\frac{\theta_2\,\omega}{r-i}\right)^{r-i}} \times \frac{p_i}{\sum_{k=1}^{r-1} p_k}\,.$$

Using the Lemma 5.A.3, we have the Theorem 5.3.2.

Chapter 6

Bayesian Analysis of Simple Step-stress Model under Weibull Lifetimes

6.1 Introduction

In the last two chapters we have discussed the frequentist and Bayesian inference of a simple step-stress model, when the lifetimes of the experimental units follow one-parameter exponential and two-parameter exponential distributions, respectively. Analysis of simple step-stress model has been performed when lifetimes have a Weibull distribution, mainly under frequentist setup. Properties of the CEM under Weibull distribution were studied in Komori [86]. Inferential aspects of step-stress model under CS-I and CS-II were addressed by Bai and Kim [6] and Kateri and Balakrishnan [83], respectively, when the distribution of lifetimes is assumed to be Weibull. Liu [100] considered step-stress model for Weibull distributed lifetimes under Bayesian setup. In all the cases it is assumed that the model satisfies CEM assumptions. However, it is noticed that MLEs of the unknown parameters do not exist in close form and therefore finding MLEs of the unknown parameters involves quite heavy computation. Most of the further statistical analysis mainly rely on asymptotic distribution of the MLEs. Moreover, extension of the analysis provide in Bai and Kim [6] and Kateri and Balakrishnan [83] are not immediate for more general censoring situations. It seems that Bayesian analysis is a natural choice in this case.

It may be worth mentioning that though some inferential issues on the parameters of Weibull distribution under step-stress model have been addressed in the literature, no attention has been paid to develop the inference imposing the order restriction on the mean lifetime at different stress levels. The frequentist approach to the order restricted inference for parameters of Weibull distribution under step-stress model is quit involved and hence, in this case also Bayesian approach is a natural alternative. In this chapter we consider a simple step-stress model, when the lifetimes are assumed to have two-parameter Weibull distribution. Though CEM is the most popular model in case of exponential lifetimes, it is not mathematically so tractable under Weibull lifetimes. Weibull CEM does not transform to exponential CEM under power transformation. Moreover, it may be worth mentioning that the KHM and CEM for Weibull distributed lifetimes can be difficult to distinguish, see Khamis and Higgins [85]. For these reasons analysis in this chapter has been performed under KHM assumptions, which is mathematically more tractable than CEM assumptions.

Rest of the chapter is organized as follows. Model assumptions and prior information on the unknown parameters are considered in Section 6.2. In Section 6.3, we provide the posterior analysis and the Bayes estimators in details for Type-I censored data. In Section 6.4, simulation study has been performed to judge the effectiveness of the procedures described in Section 6.3. In the same section we provide data analysis to illustrate the procedures proposed in Section 6.3. In Section 6.5, we have indicated how the proposed method can be implemented for other censoring schemes. Finally, the chapter is concluded in Section 6.6.

6.2 Model Assumption and Prior Information

It is assumed that the lifetime of the experimental units are independently distributed random variables having Weibull distribution. PDF and the CDF of the lifetime under stress level s_i for $i = 1, 2$, is given by

$$f(t;\, \beta,\, \lambda_i) = \beta\lambda_i t^{\beta-1}\, e^{-\lambda_i t^{\beta}} \qquad \text{for} \quad 0 < t < \infty \qquad \beta > 0 \qquad \lambda_i > 0 \tag{6.1}$$

and

$$F(t;\, \beta,\, \lambda_i) = 1 - e^{-\lambda_i t^{\beta}} \qquad \text{for} \quad 0 < t < \infty \qquad \beta > 0 \qquad \lambda_i > 0, \tag{6.2}$$

respectively. It is further assumed that the failure time data come from a KHM under the step-stress pattern, hence, it has the following CDF;

$$G(t;\, \beta\, \lambda_1,\, \lambda_2) = \begin{cases} 1 - e^{\lambda_1 t^{\beta}} & \text{if } 0 < t < \tau_1 \\ \\ 1 - e^{-\lambda_2 (t^{\beta} - \tau_1^{\beta}) - \lambda_1 \tau_1^{\beta}} & \text{if } \tau_1 \le t < \infty. \end{cases} \tag{6.3}$$

The corresponding PDF is given by

$$g(t;\, \beta,\, \lambda_1,\, \lambda_2) = \begin{cases} \beta\lambda_1\, t^{\beta-1}\, e^{-\lambda_1 t^{\beta}} & \text{if } 0 < t < \tau_1 \\ \\ \beta\lambda_2\, t^{\beta-1}\, e^{-\lambda_2 (t^{\beta} - \tau_1^{\beta}) - \lambda_1 \tau_1^{\beta}} & \text{if } \tau_1 \le t < \infty. \end{cases} \tag{6.4}$$

For developing the Bayesian inference, we need to assume some priors on the unknown parameters. If β is known, λ_1 and λ_2 have conjugate gamma priors. However, following the argument of Soland [128] it can be shown that there does not exist any continuous conjugate prior for $(\beta,\, \lambda_1,\, \lambda_2)$. A continuous-discrete conjugate prior do exist, where continuous part corresponds to the scale parameters and discrete part corresponds to the shape parameter. Kaminskiy and Krivtsov [82] criticized this choice of priors in case of complete sample constant stress life test, for it is difficulty to apply in real life and thus is not addressed further.

Following the approach of Berger and Sun [40], Kundu and Gupta [91], and Kundu [88], here we assume that λ_i, $i = 1,\, 2$, has a gamma prior with shape and scale parameters $a_i > 0$ and $b_i > 0$, respectively, *i.e.*, the prior assumption on λ_i is summarized in the following PDF

$$\pi_i(\lambda_i) \propto \lambda_i^{a_i - 1}\, e^{-\lambda_i b_i} \quad \text{for} \quad \lambda_i > 0. \tag{6.5}$$

The prior on the shape parameter β is also assumed to be a gamma distribution with shape and scale parameter $a_3 > 0$ and $b_3 > 0$, respectively, *i.e.*, the prior density of β is given by

$$\pi_3(\beta) \propto \beta^{a_3 - 1}\, e^{-b_3 \beta} \quad \text{for} \quad \beta > 0. \tag{6.6}$$

It is further assumed that β, λ_1, and λ_2 are independently distributed. We discuss the posterior analysis of Type-I censored data in details in Section 6.3.1 under this prior assumptions.

Next we consider order restricted inference of the parameters under the same model assumptions. Note that the main aim of a SSLT is to get rapid failures by imposing severe stress level on the product under test. Hence, it is natural to assume that the mean lifetime at the stress level s_1 is greater than that at the stress level s_2, which implies $\lambda_1 < \lambda_2$ under lifetime distribution (6.2). Therefore, $\lambda_1 = \alpha\lambda_2$ with $0 < \alpha < 1$. The following priors are assumed under this order restricted situation. It is assumed that priors on β and λ_2 are same as the previous case, *i.e.*, they have priors $\pi_2(\cdot)$ and $\pi_3(\cdot)$, respectively, and α has a beta prior, with parameters $a_4 > 0$ and $b_4 > 0$, having PDF

$$\pi_4(\alpha) \propto \alpha^{a_4 - 1}\, (1 - \alpha)^{b_4 - 1} \quad \text{for} \quad 0 < \alpha < 1. \tag{6.7}$$

Here also we assume that α, β, and θ_2 are independently distributed. Note that the joint prior density of (λ_1, λ_2) is same as given in (4.7). A detailed discussion of the posterior analysis of Type-I censored data under this prior assumption is provided in Section 6.3.2.

6.3 Posterior Analysis under CS-I

6.3.1 Under Unrestricted Prior Assumption

Recall that the form of the ordered observed data under CS-I can have one of the following forms:

(a) $\tau_1 < t_{1:n} < \ldots < t_{n_2:n} < \tau_2$,

(b) $t_{1:n} < \ldots < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < \ldots < t_{n_1+n_2:n} < \tau_2$,

(c) $t_{1:n} < \ldots < t_{n_1:n} < \tau_1 < \tau_2$.

Let n_1^* and n_2^* be the number of failures at the stress level s_1 and s_2, respectively, and τ^* be the experimental termination time. In case of CS-I, $\tau^* = \tau_2$. For Case (a): $n_1^* = 0$, $n_2^* = n_2 \leq n$, Case (b): $n_1^* = n_1 > 0$, $n_2^* = n_2 > 0$, Case (c): $n_1^* = n_1 > 0$, $n_2^* = 0$. Let $n^* = n_1^* + n_2^*$. Based on the observations from a simple SSLT under Type-I censoring scheme, the likelihood function can be written as

$$l_1(\text{Data}\,|\,\beta,\,\lambda_1,\,\lambda_2) \propto \beta^{n_1^*+n_2^*}\,\lambda_1^{n_1^*}\,\lambda_2^{n_2^*}\left(\prod_{i=1}^{n_1^*+n_2^*} t_{i:n}\right)^{\beta-1} e^{-\lambda_1 D_1(\beta)-\lambda_2 D_2(\beta)}, \tag{6.8}$$

where $D_1(\beta) = \sum_{j=1}^{n_1^*} t_{j:n}^{\beta} + (n - n_1^*)\tau_1^{\beta}$ and $D_2(\beta) = \sum_{j=n_1^*+1}^{n^*}(t_{j:n}^{\beta} - \tau_1^{\beta}) + (n - n^*)(\tau^{*\beta} - \tau_1^{\beta})$. Therefore, based on the prior $\pi_1(\cdot)$, $\pi_2(\cdot)$, and $\pi_3(\cdot)$ mentioned above posterior density function of β, λ_1, and λ_2 becomes

$$l_2(\beta,\,\lambda_1,\,\lambda_2\,|\,\text{Data}) \propto \beta^{n^*+a_3-1}\,\lambda_1^{n_1^*+a_1-1}\,\lambda_2^{n_2+a_2-1}\,e^{-(b_3-c_1)\,\beta-\lambda_1\,A_1(\beta)-\lambda_2\,A_2(\beta)}$$
$$\text{if}\quad \beta > 0,\ \lambda_1 > 0,\ \lambda_2 > 0, \tag{6.9}$$

where $A_1(\beta) = b_1 + D_1(\beta)$, $A_2(\beta) = b_2 + D_2(\beta)$, and $c_1 = \sum_{i=1}^{n^*} \ln t_{i:n}$. Note that the right hand side of (6.9) is integrable if we take proper priors on the unknown parameters, see Appendix 6.A.1 for details. If we want to compute the Bayes estimate of some function of β, λ_1, and λ_2, say $g(\beta,\,\lambda_1,\,\lambda_2)$, with respect to the squared error loss function, it will be posterior expectation of $g(\beta,\,\lambda_1,\,\lambda_2)$, *i.e.*,

$$\widehat{g}(\beta,\,\lambda_1,\,\lambda_2) = \int_0^\infty\int_0^\infty\int_0^\infty g(\beta,\,\lambda_1,\,\lambda_2) l_2(\beta,\,\lambda_1,\,\lambda_2\,|\,\text{Data})\,d\lambda_2 d\lambda_1 d\beta. \tag{6.10}$$

Unfortunately, the close form of (6.10) cannot be obtained in most of the cases. One may use numerical techniques to compute (6.10). Alternatively, other approximation can be used to compute (6.10). However, CRI for a parametric function cannot be constructed by these numerical methods. Hence, we propose to use importance sampling to compute Bayes estimate as well as to construct CRI of a parametric function. Note that

$$l_2(\beta,\, \lambda_1,\, \lambda_2 \,|\, \text{Data}) = l_3(\lambda_1,\, |\, \beta,\, \text{Data}) \times l_4(\lambda_2,\, |\, \beta,\, \text{Data}) \times l_5(\beta \,|\, \text{Data}), \qquad (6.11)$$

where

$$l_3(\lambda_1,\, |\, \beta,\, \text{Data}) = \frac{\{A_1(\beta)\}^{n_1^*+a_1}}{\Gamma(n_1^*+a_1)}\, \lambda_1^{n_1^*+a_1-1}\, e^{-\lambda_1\, A_1(\beta)} \quad \text{if} \quad \lambda_1 > 0, \qquad (6.12)$$

$$l_4(\lambda_2,\, |\, \beta,\, \text{Data}) = \frac{\{A_2(\beta)\}^{n_2^*+a_2}}{\Gamma(n_2^*+a_2)}\, \lambda_2^{n_2^*+a_2-1}\, e^{-\lambda_2\, A_2(\beta)} \quad \text{if} \quad \lambda_2 > 0, \qquad (6.13)$$

and

$$l_5(\beta \,|\, \text{Data}) = c_2 \frac{\beta^{n^*+a_3-1} e^{-(b_3-c_1)\beta}}{\{A_1(\beta)\}^{n_1^*+a_1}\{A_2(\beta)\}^{n_2^*+a_2}} \quad \text{if} \quad \beta > 0. \qquad (6.14)$$

The normalizing constant c_2 in (6.14) can be found using numerical method. Though it is not easy to prove the log-concavity of the $l_5(\beta \,|\, \text{Data})$, the plots (see Figure 6.1) suggest that $l_5(\beta \,|\, \text{Data})$ is a unimodal function. Hence, we try to approximate $l_5(\beta \,|\, \text{Data})$ by a gamma density function, where the parameters of the gamma distribution are determined by equating mean and variance of $l_5(\beta \,|\, \text{Data})$ to those of a gamma distribution. Let m_1 and m_2 denote the mean and variance, respectively, corresponding to the density $l_5(\beta \,|\, \text{Data})$. The shape and scale parameters of the approximating gamma distribution are given by $a_5 = \dfrac{m_1^2}{m_2}$ and $b_5 = \dfrac{m_1}{m_2}$, respectively. Let us define

$$l_6(\beta \,|\, \text{Data}) = \frac{b_5^{a_5}}{\Gamma(a_5)}\, \beta^{a_5-1}\, e^{-b_5\beta} \quad \text{for} \quad \beta > 0.$$

Note that $l_2(\beta,\, \lambda_1,\, \lambda_2 \,|\, \text{Data})$ can be expressed as follows.

$$l_2(\beta,\, \lambda_1,\, \lambda_2 \,|\, \text{Data}) = w_1(\beta) \times l_3(\lambda_1,\, |\, \beta,\, \text{Data}) \times l_4(\lambda_2,\, |\, \beta,\, \text{Data}) \times l_6(\beta \,|\, \text{Data}),$$

where $w_1(\beta) = \dfrac{l_5(\beta \mid \text{Data})}{l_6(\beta \mid \text{Data})}$. Now we propose to use the following algorithm based on importance sampling technique to compute Bayes estimate and to construct the CRI of some function $g(\beta, \lambda_1, \lambda_2)$.

Algorithm 6.3.1

Step 1. Generate β_1 from Gamma(a_5, b_5) distribution.

Step 2. For the given β_1, generate λ_{11} from (6.12).

Step 3. For the given β_1, generate λ_{21} from (6.13).

Step 4. Continue the process M times to get $\{(\beta_1, \lambda_{11}, \lambda_{21}), \ldots, (\beta_M, \lambda_{1M}, \lambda_{2M})\}$.

Step 5. Compute $g_i = g(\beta_i, \lambda_{1i}, \lambda_{2i});\ i = 1, 2, \ldots, M$.

Step 6. Calculate the weights $w_{1i} = w_1(\beta_i);\ i = 1, 2, \ldots, M$.

Step 7. Compute the BE of $g(\beta, \lambda_1, \lambda_2)$ as

$$\widehat{g}_{BE}(\beta, \lambda_1, \lambda_2) = \frac{1}{M}\sum_{j=1}^{M} w_{1j} g_j.$$

Step 8. To construct a $100(1-\gamma)\%$ CRI of $g(\beta, \lambda_1, \lambda_2)$, first order g_j for $j = 1, \ldots, M$, say $g_{(1)} < g_{(2)} < \ldots < g_{(M)}$, and order w_j accordingly to get $w_{1(1)}, w_{1(2)}, \ldots, w_{1(M)}$. Note that $w_{1(1)}, w_{1(2)}, \ldots, w_{1(M)}$ may not be ordered. A $100(1-\gamma)\%$ CRI can be obtained as $(g_{(j_1)}, g_{(j_2)})$, where j_1 and j_2 satisfy

$$j_1, j_2 \in \{1, 2, \ldots, M\}, \quad j_1 < j_2, \quad \frac{1}{M}\sum_{i=j_1}^{j_2} w_{1(i)} \leq 1 - \gamma < \frac{1}{M}\sum_{i=j_1}^{j_2+1} w_{1(i)}. \tag{6.15}$$

The $100(1-\gamma)\%$ HPD CRI of $g(\beta, \lambda_1, \lambda_2)$ becomes $(g_{(j_1^*)}, g_{(j_2^*)})$, where $j_1^* < j_2^*$, $j_1^*, j_2^* \in \{1, 2, \ldots, M\}$ satisfy

$$\frac{1}{M}\sum_{i=j_1^*}^{j_2^*} w_{1(i)} \leq 1 - \gamma < \frac{1}{M}\sum_{i=j_1^*}^{j_2^*+1} w_{1(i)}, \qquad g_{(j_2^*)} - g_{(j_1^*)} \leq g_{(j_2)} - g_{(j_1)},$$

for all j_1 and j_2 satisfying (6.15).

6.3.2 Under Order Restricted Prior Assumption

Computations of Bayes estimate and construction of associated CRI of some parametric function $g(\beta,\, \lambda_1,\, \lambda_2)$ under order restricted priors are addressed in this subsection. Using the reparameterization $\lambda_1 = \alpha\, \lambda_2$ $(0 < \alpha < 1)$ and (6.5), (6.6), (6.7), and (6.8), one can express the posterior density function of $(\alpha,\, \beta,\, \lambda_2)$ as

$$l_7(\alpha,\, \beta,\, \lambda_2 \,|\, \text{Data}) \propto \alpha^{n_1^*+a_4-1}(1-\alpha)^{b_4-1}\beta^{n^*+a_3-1}\lambda_2^{n^*+a_2-1} \times e^{-\lambda_2(\alpha D_1(\beta)+D_2(\beta)+b_2)-(b_3-c_1)\beta} \quad \text{if } 0<\alpha<1,\ \beta>0,\ \lambda_2>0. \tag{6.16}$$

Like the previous case, the right hand side of (6.16) is integrable if proper priors are assumed on the unknown parameters, see Appendix 6.A.2 for details. Now under squared error loss function Bayes estimate of some parametric function $g(\alpha,\, \beta,\, \lambda_2)$ is given by

$$\widehat{g}(\alpha,\, \beta,\, \lambda_2) = \int_0^1\int_0^\infty\int_0^\infty g(\alpha,\, \beta,\, \lambda_2) l_7(\alpha,\, \beta,\, \lambda_2 \,|\, \text{Data})\, d\lambda_2 d\beta d\alpha. \tag{6.17}$$

Note that

$$l_7(\alpha,\, \beta,\, \lambda_2 \,|\, \text{Data}) \propto w_2(\alpha,\, \beta) \times l_8(\lambda_2 \,|\, \alpha,\, \beta,\, \text{Data}) \times l_9(\beta \,|\, \text{Data}), \tag{6.18}$$

where

$$w_2(\alpha,\, \beta) = \frac{\alpha^{n_1^*+a_4-1}(1-\alpha)^{b_4-1}}{\{\alpha D_1(\beta)+D_2(\beta)+b_2\}^{n^*+a_2}}, \tag{6.19}$$

$$l_8(\lambda_2 \,|\, \alpha,\, \beta,\, \text{Data}) = \frac{\{\alpha D_1(\beta)+D_2(\beta)+b_2\}^{n^*+a_2}}{\Gamma(n^*+a_2)}\lambda_2^{n^*+a_2-1}\, e^{-\lambda_2(\alpha D_1(\beta)+D_2(\beta)+b_2)}, \tag{6.20}$$

and

$$l_9(\beta \,|\, \text{Data}) = \frac{(b_3-c_1)^{n^*+a_3}}{\Gamma(n^*+a_3)}\beta^{n^*+a_3-1}\, e^{-(b_3-c_1)\beta}. \tag{6.21}$$

Depending upon the previous expression of $l_7(\alpha,\, \beta,\, \lambda_2 \,|\, \text{Data})$, the following algorithm is proposed to compute Bayes estimate as well as to construct CRI.

Algorithm 6.3.2

Step 1. Generate α_1 from U(0, 1) distribution.

Step 2. Generate β_1 from (6.21).

Step 3. For the given α_1 and β_1, generate λ_{21} from (6.20).

Step 4. Continue the process M times to get $\{(\alpha_1, \beta_1, \lambda_{21}), \ldots, (\alpha_M, \beta_M, \lambda_{2M})\}$.

Step 5. Calculate $g_i = g(\alpha_i, \beta_i, \lambda_{2i});\ i = 1, 2, \ldots, M$.

Step 6. Calculate the weights $w_{2i} = w_2(\alpha_i, \beta_i);\ i = 1, 2, \ldots, M$.

Step 7. Calculate the normalize weights $w^*_{2i} = \dfrac{w_{2i}}{\sum_{j=1}^{M} w_{2j}};\ i = 1, 2, \ldots, M$.

Step 8. Compute the BE of $g(\alpha, \beta, \lambda_2)$ as $\widehat{g}_{BE}(\beta, \lambda_1, \lambda_2) = \displaystyle\sum_{j=1}^{M} w^*_{2j} g_j$.

Step 9. To construct a $100(1-\gamma)\%$ CRI of $g(\alpha, \beta, \lambda_2)$, first order g_j for $j = 1, \ldots, M$, say $g_{(1)} < g_{(2)} < \ldots < g_{(M)}$, and order w^*_{2j} accordingly to get $w^*_{2(1)}, w^*_{2(2)}, \ldots, w^*_{2(M)}$. Note that $w^*_{2(1)}, w^*_{2(2)}, \ldots, w^*_{2(M)}$ may not be ordered. A $100(1-\gamma)\%$ CRI can be obtained as $(g_{(j_1)}, g_{(j_2)})$, where j_1 and j_2 satisfy

$$j_1, j_2 \in \{1, 2, \ldots, M\}, \quad j_1 < j_2, \quad \sum_{i=j_1}^{j_2} w^*_{2(i)} \le 1-\gamma < \sum_{i=j_1}^{j_2+1} w^*_{2(i)}. \tag{6.22}$$

The $100(1-\gamma)\%$ HPD CRI of $g(\alpha, \beta, \lambda_2)$ becomes $(g_{(j_1^*)}, g_{(j_2^*)})$, where $j_1^* < j_2^*$, $j_1^*, j_2^* \in \{1, 2, \ldots, M\}$ satisfy

$$\sum_{i=j_1^*}^{j_2^*} w^*_{2(i)} \le 1-\gamma < \sum_{i=j_1^*}^{j_2^*+1} w^*_{2(i)}, \qquad g_{(j_2^*)} - g_{(j_1^*)} \le g_{(j_2)} - g_{(j_1)},$$

for all j_1 and j_2 satisfying (6.22).

6.4 Simulations and Data Analysis

6.4.1 Simulation Results

In this section we present some simulation results to judge how the proposed procedures work for different values of τ_1, τ_2 and n. Here we choose $\beta = 2$, $\lambda_1 = 1/1.2 \simeq$

Table 6.1: AE and MSE of BE of β, λ_1, and λ_2 for unrestricted case with $a_1 = 0.0001$, $b_1 = 0.0001$, $a_2 = 0.0001$, $b_2 = 0.0001$, $a_3 = 0.0001$, $b_3 = 0.0001$, $\beta = 2$, $\lambda_1 = 0.833$, and $\lambda_2 = 2.222$.

			β		λ_1		λ_2	
n	τ_1	τ_2	AE	MSE	AE	MSE	AE	MSE
40	0.60	0.80	2.180	0.5033	1.087	0.4869	2.477	0.5826
		1.20	2.105	0.3179	0.981	0.2372	2.429	0.4292
	0.65	0.85	2.159	0.4464	0.997	0.2378	2.438	0.5955
		1.25	2.099	0.2961	0.942	0.1559	2.437	0.4792
	0.70	0.90	2.139	0.3707	0.939	0.1341	2.415	0.6383
		1.30	2.103	0.2742	0.918	0.1143	2.424	0.5191
	0.75	0.95	2.118	0.3147	0.902	0.0922	2.426	0.6838
		1.35	2.105	0.2601	0.899	0.0848	2.425	0.5768
	0.80	1.00	2.110	0.2934	0.881	0.0669	2.406	0.7211
		1.40	2.111	0.2478	0.886	0.0650	2.435	0.6153
50	0.60	0.80	2.153	0.4111	1.028	0.3297	2.405	0.4110
		1.20	2.071	0.2375	0.943	0.1611	2.395	0.3357
	0.65	0.85	2.136	0.3322	0.959	0.1586	2.374	0.4159
		1.25	2.085	0.2258	0.927	0.1198	2.384	0.3453
	0.70	0.90	2.112	0.2841	0.921	0.1014	2.354	0.4610
		1.30	2.082	0.2115	0.905	0.0843	2.384	0.4023
	0.75	0.95	2.101	0.2445	0.895	0.0706	2.354	0.4933
		1.35	2.091	0.2005	0.889	0.0638	2.369	0.4214
	0.80	1.00	2.084	0.2096	0.874	0.0525	2.367	0.5525
		1.40	2.091	0.1890	0.876	0.0495	2.358	0.4355

Table 6.2: CP and AL of Symmetric CRI of β for unrestricted case with $a_1 = 0.0001$, $b_1 = 0.0001$, $a_2 = 0.0001$, $b_2 = 0.0001$, $a_3 = 0.0001$, $b_3 = 0.0001$, $\beta = 2$, $\lambda_1 = 0.833$, and $\lambda_2 = 2.222$.

			90%		95%		99%	
n	τ_1	τ_2	CP	AL	CP	AL	CP	AL
40	0.60	0.80	90.42	2.124	95.54	2.534	99.28	3.340
		1.20	90.04	1.753	94.70	2.084	98.86	2.721
	0.65	0.85	89.86	1.966	95.16	2.346	99.10	3.095
		1.25	90.22	1.694	94.94	2.014	98.96	2.634
	0.70	0.90	90.10	1.828	95.10	2.181	99.12	2.876
		1.30	89.84	1.638	95.10	1.949	98.88	2.554
	0.75	0.95	90.40	1.707	94.92	2.037	99.10	2.686
		1.35	90.32	1.578	95.42	1.878	99.18	2.464
	0.80	1.00	89.24	1.608	94.52	1.919	98.82	2.530
		1.40	90.04	1.522	94.68	1.813	98.98	2.380
50	0.60	0.80	89.68	1.877	94.78	2.239	99.02	2.950
		1.20	89.76	1.552	95.26	1.845	99.02	2.413
	0.65	0.85	90.22	1.738	95.00	2.073	99.06	2.732
		1.25	89.60	1.506	95.02	1.792	99.14	2.346
	0.70	0.90	89.80	1.609	94.96	1.920	99.12	2.530
		1.30	90.18	1.450	95.18	1.726	98.86	2.261
	0.75	0.95	90.20	1.508	95.06	1.799	98.96	2.371
		1.35	90.10	1.402	95.16	1.670	98.92	2.191
	0.80	1.00	89.46	1.418	95.12	1.691	99.06	2.227
		1.40	90.38	1.349	94.98	1.607	99.08	2.111

Table 6.3: CP and AL of HPD CRI of β for unrestricted case with $a_1 = 0.0001$, $b_1 = 0.0001$, $a_2 = 0.0001$, $b_2 = 0.0001$, $a_3 = 0.0001$, $b_3 = 0.0001$, $\beta = 2$, $\lambda_1 = 0.833$, and $\lambda_2 = 2.222$.

			90%		95%		99%	
n	τ_1	τ_2	CP	AL	CP	AL	CP	AL
40	0.60	0.80	89.30	2.077	95.22	2.477	99.16	3.257
		1.20	89.16	1.734	94.24	2.059	98.64	2.678
	0.65	0.85	89.78	1.927	95.06	2.298	98.98	3.025
		1.25	89.40	1.673	94.30	1.989	98.78	2.593
	0.70	0.90	90.30	1.795	94.86	2.140	98.88	2.816
		1.30	89.10	1.618	94.54	1.925	98.84	2.515
	0.75	0.95	89.78	1.679	94.92	2.003	98.84	2.635
		1.35	90.00	1.559	94.92	1.855	99.12	2.427
	0.80	1.00	89.20	1.584	94.28	1.890	98.88	2.484
		1.40	89.92	1.504	94.64	1.791	98.90	2.345
50	0.60	0.80	89.48	1.843	94.62	2.197	98.78	2.888
		1.20	89.28	1.536	94.52	1.826	98.82	2.380
	0.65	0.85	89.98	1.709	94.68	2.038	98.98	2.679
		1.25	89.04	1.491	94.28	1.773	99.02	2.314
	0.70	0.90	89.92	1.585	94.58	1.890	99.16	2.485
		1.30	89.84	1.435	95.02	1.707	98.78	2.231
	0.75	0.95	89.92	1.488	94.94	1.774	98.84	2.332
		1.35	89.70	1.388	94.90	1.652	98.92	2.162
	0.80	1.00	89.62	1.400	94.82	1.669	98.88	2.193
		1.40	90.02	1.336	94.92	1.590	99.08	2.083

Table 6.4: CP and AL of Symmetric CRI of λ_1 for unrestricted case with $a_1 = 0.0001$, $b_1 = 0.0001$, $a_2 = 0.0001$, $b_2 = 0.0001$, $a_3 = 0.0001$, $b_3 = 0.0001$, $\beta = 2$, $\lambda_1 = 0.833$, and $\lambda_2 = 2.222$.

			90%		95%		99%	
n	τ_1	τ_2	CP	AL	CP	AL	CP	AL
40	0.60	0.80	90.52	1.803	95.02	2.322	98.90	3.712
		1.20	89.40	1.392	94.56	1.723	98.86	2.482
	0.65	0.85	89.92	1.384	94.74	1.734	99.12	2.584
		1.25	89.88	1.183	94.94	1.451	98.98	2.043
	0.70	0.90	90.60	1.115	95.22	1.370	99.20	1.945
		1.30	89.14	1.028	94.90	1.251	98.98	1.730
	0.75	0.95	89.82	0.937	94.92	1.138	98.84	1.569
		1.35	89.66	0.905	94.70	1.094	99.04	1.490
	0.80	1.00	89.90	0.820	95.12	0.989	98.88	1.339
		1.40	90.30	0.809	95.44	0.974	99.12	1.312
50	0.60	0.80	89.64	1.497	94.96	1.889	98.78	2.867
		1.20	89.16	1.187	94.56	1.456	98.54	2.057
	0.65	0.85	90.42	1.171	95.38	1.447	98.96	2.083
		1.25	89.64	1.035	94.48	1.262	99.18	1.751
	0.70	0.90	89.42	0.965	94.50	1.177	98.64	1.639
		1.30	89.86	0.901	94.40	1.092	98.74	1.493
	0.75	0.95	89.40	0.826	94.48	0.999	98.70	1.362
		1.35	89.86	0.799	95.34	0.963	98.78	1.302
	0.80	1.00	89.22	0.723	94.70	0.870	98.86	1.170
		1.40	90.24	0.715	95.26	0.859	99.10	1.151

Table 6.5: CP and AL of HPD CRI of λ_1 for unrestricted case with $a_1 = 0.0001$, $b_1 = 0.0001$, $a_2 = 0.0001$, $b_2 = 0.0001$, $a_3 = 0.0001$, $b_3 = 0.0001$, $\beta = 2$, $\lambda_1 = 0.833$, and $\lambda_2 = 2.222$.

			90%		95%		99%	
n	τ_1	τ_2	CP	AL	CP	AL	CP	AL
40	0.60	0.80	90.70	1.551	95.24	2.000	99.04	3.204
		1.20	89.08	1.270	94.22	1.575	98.84	2.275
	0.65	0.85	89.42	1.249	94.94	1.564	98.88	2.329
		1.25	90.36	1.040	95.16	1.278	99.10	1.811
	0.70	0.90	88.70	0.974	94.52	1.185	98.92	1.637
		1.30	89.68	1.102	94.74	1.353	98.80	1.906
	0.75	0.95	89.10	0.894	94.66	1.085	98.98	1.493
		1.35	89.16	0.868	94.80	1.050	98.88	1.428
	0.80	1.00	89.34	0.793	94.52	0.956	98.84	1.291
		1.40	89.40	0.784	95.22	0.944	98.98	1.268
50	0.60	0.80	90.16	1.330	94.82	1.681	99.02	2.551
		1.20	88.50	1.103	93.70	1.355	98.40	1.915
	0.65	0.85	89.94	1.082	95.16	1.337	99.14	1.923
		1.25	88.94	0.978	94.76	1.192	98.92	1.654
	0.70	0.90	89.18	0.914	94.30	1.115	98.50	1.549
		1.30	89.70	0.863	94.70	1.046	98.60	1.428
	0.75	0.95	89.22	0.796	94.52	0.962	98.62	1.308
		1.35	90.04	0.773	94.94	0.931	98.68	1.257
	0.80	1.00	88.86	0.704	94.14	0.846	98.64	1.135
		1.40	90.12	0.698	94.92	0.838	99.00	1.120

Table 6.6: CP and AL of Symmetric CRI of λ_2 for unrestricted case with $a_1 = 0.0001$, $b_1 = 0.0001$, $a_2 = 0.0001$, $b_2 = 0.0001$, $a_3 = 0.0001$, $b_3 = 0.0001$, $\beta = 2$, $\lambda_1 = 0.833$, and $\lambda_2 = 2.222$.

			90%		95%		99%	
n	τ_1	τ_2	CP	AL	CP	AL	CP	AL
40	0.60	0.80	90.00	2.393	95.08	2.919	99.06	4.097
		1.20	90.34	2.057	94.90	2.529	98.88	3.616
	0.65	0.85	89.62	2.402	94.76	2.927	98.90	4.085
		1.25	90.20	2.171	95.50	2.660	98.90	3.764
	0.70	0.90	89.10	2.455	94.18	2.989	98.76	4.158
		1.30	89.64	2.264	95.00	2.769	99.02	3.890
	0.75	0.95	89.18	2.555	94.20	3.111	98.92	4.320
		1.35	89.98	2.360	94.86	2.882	98.92	4.027
	0.80	1.00	89.14	2.628	94.60	3.198	98.92	4.428
		1.40	89.94	2.456	94.84	2.994	99.08	4.166
50	0.60	0.80	90.74	2.060	95.18	2.496	99.14	3.446
		1.20	89.60	1.798	94.80	2.193	98.96	3.068
	0.65	0.85	90.40	2.075	95.20	2.511	99.04	3.444
		1.25	90.20	1.880	94.98	2.288	98.94	3.180
	0.70	0.90	89.36	2.138	94.72	2.590	98.88	3.552
		1.30	89.88	1.977	94.70	2.403	98.98	3.324
	0.75	0.95	89.40	2.217	94.72	2.686	98.92	3.681
		1.35	89.80	2.050	94.84	2.488	98.70	3.429
	0.80	1.00	88.82	2.308	94.18	2.797	98.88	3.829
		1.40	90.20	2.122	95.34	2.575	99.10	3.538

Table 6.7: CP and AL of HPD CRI of λ_2 for unrestricted case with $a_1 = 0.0001$, $b_1 = 0.0001$, $a_2 = 0.0001$, $b_2 = 0.0001$, $a_3 = 0.0001$, $b_3 = 0.0001$, $\beta = 2$, $\lambda_1 = 0.833$, and $\lambda_2 = 2.222$.

			90%		95%		99%	
n	τ_1	τ_2	CP	AL	CP	AL	CP	AL
40	0.60	0.80	90.60	2.296	95.42	2.792	99.18	3.883
		1.20	90.30	1.957	95.56	2.399	99.18	3.399
	0.65	0.85	90.06	2.302	94.96	2.798	99.00	3.877
		1.25	90.84	2.065	95.74	2.526	99.06	3.551
	0.70	0.90	89.16	2.348	94.24	2.854	98.64	3.948
		1.30	90.20	2.152	95.10	2.628	99.00	3.676
	0.75	0.95	88.98	2.436	94.36	2.963	98.86	4.100
		1.35	89.70	2.243	94.96	2.736	99.04	3.810
	0.80	1.00	88.32	2.500	93.96	3.040	98.62	4.201
		1.40	89.80	2.333	94.76	2.842	99.04	3.945
50	0.60	0.80	91.20	1.996	95.82	2.413	99.24	3.303
		1.20	90.46	1.728	95.38	2.105	99.08	2.924
	0.65	0.85	90.96	2.008	95.20	2.428	99.10	3.311
		1.25	90.86	1.807	95.62	2.197	99.06	3.036
	0.70	0.90	89.32	2.063	94.56	2.496	99.04	3.409
		1.30	89.58	1.898	94.88	2.306	99.00	3.177
	0.75	0.95	88.56	2.134	94.18	2.584	98.80	3.529
		1.35	89.54	1.967	94.58	2.387	98.96	3.279
	0.80	1.00	88.52	2.217	93.78	2.685	98.76	3.668
		1.40	89.90	2.036	94.86	2.469	98.84	3.385

Table 6.8: AE and MSE of BE of β, λ_1, and λ_2 for order restricted case with $a_1 = 0.0001$, $b_1 = 0.0001$, $a_2 = 0.0001$, $b_2 = 0.0001$, $a_4 = 1$, $b_4 = 1$, $\beta = 2$, $\lambda_1 = 0.833$, and $\lambda_2 = 2.222$.

			β		λ_1		λ_2	
n	τ_1	τ_2	AE	MSE	AE	MSE	AE	MSE
40	0.60	0.80	2.156	0.2940	0.984	0.1186	2.302	0.3892
		1.20	2.201	0.2457	0.997	0.1057	2.250	0.2503
	0.65	0.85	2.191	0.3094	0.980	0.1040	2.272	0.3565
		1.25	2.212	0.2410	0.976	0.0872	2.219	0.2576
	0.70	0.90	2.215	0.3279	0.962	0.0866	2.233	0.3652
		1.30	2.219	0.2404	0.960	0.0727	2.185	0.2832
	0.75	0.95	2.196	0.2990	0.938	0.0668	2.226	0.4216
		1.35	2.242	0.2616	0.944	0.0604	2.168	0.3476
	0.80	1.00	2.197	0.2703	0.923	0.0565	2.186	0.4282
		1.40	2.264	0.3077	0.932	0.0507	2.134	0.3644
50	0.60	0.80	2.110	0.2156	0.958	0.0915	2.250	0.2940
		1.20	2.185	0.1990	0.985	0.0947	2.226	0.1973
	0.65	0.85	2.158	0.2430	0.961	0.0851	2.239	0.3071
		1.25	2.200	0.2048	0.965	0.0750	2.198	0.2100
	0.70	0.90	2.169	0.2376	0.943	0.0676	2.214	0.3092
		1.30	2.206	0.2089	0.948	0.0601	2.167	0.2413
	0.75	0.95	2.168	0.2304	0.922	0.0552	2.205	0.3309
		1.35	2.228	0.2169	0.937	0.0490	2.136	0.2883
	0.80	1.00	2.161	0.2086	0.911	0.0443	2.178	0.3438
		1.40	2.250	0.2350	0.926	0.0412	2.083	0.3050

Table 6.9: CP and AL of Symmetric CRI of β for order restricted case with $a_1 = 0.0001$, $b_1 = 0.0001$, $a_2 = 0.0001$, $b_2 = 0.0001$, $a_4 = 1$, $b_4 = 1$, $\beta = 2$, $\lambda_1 = 0.833$, and $\lambda_2 = 2.222$.

			90%		95%		99%	
n	τ_1	τ_2	CP	AL	CP	AL	CP	AL
40	0.60	0.80	91.12	1.573	95.36	1.763	98.98	2.037
		1.20	90.50	1.504	95.44	1.751	98.74	2.167
	0.65	0.85	91.36	1.631	96.02	1.846	99.40	2.146
		1.25	88.88	1.455	93.86	1.693	97.48	2.097
	0.70	0.90	90.12	1.656	94.92	1.905	99.20	2.257
		1.30	87.32	1.387	91.78	1.610	95.12	2.001
	0.75	0.95	90.06	1.611	95.30	1.893	99.06	2.328
		1.35	82.28	1.313	86.76	1.525	90.00	1.900
	0.80	1.00	89.98	1.533	94.84	1.817	98.92	2.323
		1.40	76.88	1.222	80.68	1.421	83.92	1.777
50	0.60	0.80	90.90	1.346	95.64	1.513	98.80	1.759
		1.20	90.14	1.359	95.04	1.581	98.56	1.950
	0.65	0.85	90.24	1.418	94.52	1.602	98.56	1.865
		1.25	88.24	1.310	93.22	1.521	96.60	1.880
	0.70	0.90	90.48	1.464	95.52	1.681	99.20	1.983
		1.30	84.36	1.237	89.50	1.433	92.94	1.775
	0.75	0.95	89.84	1.442	95.08	1.690	99.00	2.066
		1.35	79.10	1.153	83.50	1.336	87.04	1.661
	0.80	1.00	89.80	1.370	95.00	1.622	99.14	2.063
		1.40	72.82	1.059	76.32	1.229	79.46	1.538

Table 6.10: CP and AL of HPD CRI of β for order restricted case with $a_1 = 0.0001$, $b_1 = 0.0001$, $a_2 = 0.0001$, $b_2 = 0.0001$, $a_4 = 1$, $b_4 = 1$, $\beta = 2$, $\lambda_1 = 0.833$, and $\lambda_2 = 2.222$.

			90%		95%		99%	
n	τ_1	τ_2	CP	AL	CP	AL	CP	AL
40	0.60	0.80	84.06	1.438	92.12	1.636	98.38	1.948
		1.20	87.86	1.427	93.70	1.659	98.48	2.062
	0.65	0.85	83.64	1.516	92.12	1.729	98.84	2.057
		1.25	87.02	1.371	92.84	1.595	97.22	1.991
	0.70	0.90	82.84	1.565	91.12	1.801	98.22	2.163
		1.30	85.68	1.291	90.98	1.507	94.98	1.894
	0.75	0.95	87.00	1.545	91.96	1.805	98.20	2.227
		1.35	80.68	1.208	85.82	1.415	89.96	1.791
	0.80	1.00	88.36	1.485	93.56	1.750	98.52	2.220
		1.40	75.24	1.110	79.98	1.306	83.82	1.670
50	0.60	0.80	83.90	1.219	91.84	1.395	97.98	1.677
		1.20	87.68	1.284	93.56	1.493	98.22	1.850
	0.65	0.85	83.04	1.310	90.82	1.493	97.64	1.783
		1.25	86.30	1.226	91.96	1.427	96.46	1.781
	0.70	0.90	83.92	1.379	91.40	1.584	98.38	1.899
		1.30	82.94	1.142	88.58	1.332	92.64	1.676
	0.75	0.95	85.72	1.382	91.56	1.610	98.20	1.973
		1.35	77.74	1.050	82.78	1.231	86.84	1.562
	0.80	1.00	88.64	1.326	93.38	1.561	98.12	1.970
		1.40	71.40	0.950	75.68	1.121	79.50	1.440

Table 6.11: CP and AL of Symmetric CRI of λ_1 for order restricted case with $a_1 = 0.0001$, $b_1 = 0.0001$, $a_2 = 0.0001$, $b_2 = 0.0001$, $a_4 = 1$, $b_4 = 1$, $\beta = 2$, $\lambda_1 = 0.833$, and $\lambda_2 = 2.222$.

			90%		95%		99%	
n	τ_1	τ_2	CP	AL	CP	AL	CP	AL
40	0.60	0.80	93.36	1.107	97.42	1.291	99.58	1.604
		1.20	92.48	1.080	96.90	1.274	99.74	1.636
	0.65	0.85	92.14	1.036	96.70	1.215	99.48	1.525
		1.25	92.06	0.973	96.40	1.151	99.50	1.488
	0.70	0.90	91.98	0.945	96.36	1.117	99.30	1.426
		1.30	91.82	0.879	96.44	1.041	99.58	1.352
	0.75	0.95	91.42	0.850	96.08	1.010	99.44	1.313
		1.35	91.26	0.796	95.82	0.946	99.42	1.233
	0.80	1.00	91.48	0.767	96.16	0.914	99.06	1.199
		1.40	91.00	0.727	95.62	0.866	99.22	1.131
50	0.60	0.80	92.50	0.967	96.64	1.128	99.62	1.411
		1.20	91.60	0.993	96.38	1.171	99.48	1.506
	0.65	0.85	91.32	0.915	96.30	1.074	99.36	1.352
		1.25	91.52	0.893	96.22	1.057	99.50	1.365
	0.70	0.90	91.52	0.846	96.24	1.000	99.28	1.279
		1.30	91.58	0.799	96.00	0.948	99.32	1.233
	0.75	0.95	91.32	0.764	96.00	0.910	99.40	1.181
		1.35	91.24	0.721	96.02	0.857	99.28	1.119
	0.80	1.00	91.42	0.693	95.98	0.825	99.30	1.081
		1.40	90.58	0.653	95.44	0.777	99.10	1.017

Table 6.12: CP and AL of HPD CRI of λ_1 for order restricted case with $a_1 = 0.0001$, $b_1 = 0.0001$, $a_2 = 0.0001$, $b_2 = 0.0001$, $a_4 = 1$, $b_4 = 1$, $\beta = 2$, $\lambda_1 = 0.833$, and $\lambda_2 = 2.222$.

			90%		95%		99%	
n	τ_1	τ_2	CP	AL	CP	AL	CP	AL
40	0.60	0.80	90.34	1.025	95.38	1.212	99.22	1.537
		1.20	92.14	1.035	96.66	1.227	99.46	1.584
	0.65	0.85	89.76	0.970	94.66	1.149	99.16	1.463
		1.25	92.14	0.935	96.22	1.111	99.44	1.444
	0.70	0.90	90.10	0.897	94.84	1.065	98.82	1.372
		1.30	91.94	0.845	96.72	1.006	99.50	1.314
	0.75	0.95	90.42	0.818	95.28	0.973	99.16	1.268
		1.35	91.46	0.765	96.10	0.913	99.32	1.199
	0.80	1.00	91.50	0.746	95.82	0.889	98.96	1.164
		1.40	91.06	0.696	95.90	0.834	99.20	1.098
50	0.60	0.80	89.54	0.899	94.56	1.063	99.26	1.353
		1.20	91.12	0.952	95.98	1.129	99.30	1.459
	0.65	0.85	88.20	0.861	94.44	1.017	98.96	1.298
		1.25	91.36	0.858	96.36	1.020	99.40	1.326
	0.70	0.90	89.62	0.805	94.74	0.955	98.94	1.231
		1.30	91.30	0.767	96.10	0.915	99.12	1.198
	0.75	0.95	89.94	0.737	95.14	0.877	99.14	1.142
		1.35	91.38	0.691	96.26	0.826	99.22	1.087
	0.80	1.00	91.18	0.674	95.66	0.804	99.06	1.052
		1.40	90.44	0.622	95.48	0.746	99.02	0.986

Table 6.13: CP and AL of Symmetric CRI of λ_2 for order restricted case with $a_1 = 0.0001$, $b_1 = 0.0001$, $a_2 = 0.0001$, $b_2 = 0.0001$, $a_4 = 1$, $b_4 = 1$, $\beta = 2$, $\lambda_1 = 0.833$, and $\lambda_2 = 2.222$.

			90%		95%		99%	
n	τ_1	τ_2	CP	AL	CP	AL	CP	AL
40	0.60	0.80	90.00	2.393	95.08	2.919	99.06	4.097
		1.20	90.34	2.057	94.90	2.529	98.88	3.616
	0.65	0.85	89.62	2.402	94.76	2.927	98.90	4.085
		1.25	89.10	2.455	94.18	2.989	98.76	4.158
	0.70	0.90	89.64	2.264	95.00	2.769	99.02	3.890
		1.30	90.20	2.171	95.50	2.660	98.90	3.764
	0.75	0.95	89.18	2.555	94.20	3.111	98.92	4.320
		1.35	89.98	2.360	94.86	2.882	98.92	4.027
	0.80	1.00	89.14	2.628	94.60	3.198	98.92	4.428
		1.40	89.94	2.456	94.84	2.994	99.08	4.166
50	0.60	0.80	90.74	2.060	95.18	2.496	99.14	3.446
		1.20	89.60	1.798	94.80	2.193	98.96	3.068
	0.65	0.85	90.40	2.075	95.20	2.511	99.04	3.444
		1.25	90.20	1.880	94.98	2.288	98.94	3.180
	0.70	0.90	89.36	2.138	94.72	2.590	98.88	3.552
		1.30	89.88	1.977	94.70	2.403	98.98	3.324
	0.75	0.95	89.40	2.217	94.72	2.686	98.92	3.681
		1.35	89.80	2.050	94.84	2.488	98.70	3.429
	0.80	1.00	88.82	2.308	94.18	2.797	98.88	3.829
		1.40	90.20	2.122	95.34	2.575	99.10	3.538

Table 6.14: CP and AL of HPD CRI of λ_2 for order restricted case with $a_1 = 0.0001$, $b_1 = 0.0001$, $a_2 = 0.0001$, $b_2 = 0.0001$, $a_4 = 1$, $b_4 = 1$, $\beta = 2$, $\lambda_1 = 0.833$, and $\lambda_2 = 2.222$.

			90%		95%		99%	
n	τ_1	τ_2	CP	AL	CP	AL	CP	AL
40	0.60	0.80	90.60	2.296	95.42	2.792	99.18	3.883
		1.20	90.30	1.957	95.56	2.399	99.18	3.399
	0.65	0.85	90.06	2.302	94.96	2.798	99.00	3.877
		1.25	89.16	2.348	94.24	2.854	98.64	3.948
	0.70	0.90	90.20	2.152	95.10	2.628	99.00	3.676
		1.30	90.84	2.065	95.74	2.526	99.06	3.551
	0.75	0.95	88.98	2.436	94.36	2.963	98.86	4.100
		1.35	89.70	2.243	94.96	2.736	99.04	3.810
	0.80	1.00	88.32	2.500	93.96	3.040	98.62	4.201
		1.40	89.80	2.333	94.76	2.842	99.04	3.945
50	0.60	0.80	91.20	1.996	95.82	2.413	99.24	3.303
		1.20	90.46	1.728	95.38	2.105	99.08	2.924
	0.65	0.85	90.96	2.008	95.20	2.428	99.10	3.311
		1.25	90.86	1.807	95.62	2.197	99.06	3.036
	0.70	0.90	89.32	2.063	94.56	2.496	99.04	3.409
		1.30	89.58	1.898	94.88	2.306	99.00	3.177
	0.75	0.95	88.56	2.134	94.18	2.584	98.80	3.529
		1.35	89.54	1.967	94.58	2.387	98.96	3.279
	0.80	1.00	88.52	2.217	93.78	2.685	98.76	3.668
		1.40	89.90	2.036	94.86	2.469	98.84	3.385

0.833, and $\lambda_2 = 1/0.45 \backsimeq 2.222$. All the results are based on 5000 simulations and $M = 8000$. We also choose $a_1 = 0.0001$, $b_1 = 0.0001$, $a_2 = 0.0001$, $b_2 = 0.0001$, $a_3 = 0.0001$, $b_3 = 0.0001$, $a_4 = 1$, and $b_4 = 1$. The priors on β, λ_1 and λ_2 are assumed to be very flat in unrestricted case and they are 'almost' non-informative. Once again in order restricted case, priors on β and λ_2 are 'almost' non-informative, whereas the prior on α is a non-informative. For different values of τ_1, τ_2, and n, AEs and MSEs of BE for β, λ_1, and λ_2 are presented in Table 6.1 for unrestricted case and in Table 6.8 for order restricted case. The CPs and ALs of symmetric CRI and HPD CRI of β, λ_1, and λ_2 are reported in Tables 6.2, 6.3, 6.4, 6.5, 6.6, and 6.7, respectively, for unrestricted case and in Tables 6.9, 6.10, 6.11, 6.12, 6.13, and 6.14, respectively, for order restricted case. In all the calculations we discard those samples for which BE of any of the parameters is greater than ten times of its original value. We have noticed that for both values of n, there is only one sample for which BE of λ_1 is greater than 8.33 in case of unrestricted inference, when $\tau_1 = 0.6$ and $\tau_2 = 0.8$.

The following points are quite clear from these tables. In the unrestricted case, MSEs of all the unknown parameters decrease as n increases. As τ_1 increases, MSEs of β and λ_1 decrease. MSEs of λ_2 decrease as τ_2 increases keeping τ_1 fixed under unrestricted framework. In the sane case MSEs of λ_1 also decrease with increase in τ_2. CPs of symmetric and HPD CRI maintain its nominal level for all the parameters. It is noticed that ALs of symmetric and HPD CRI for all unknown parameters decrease as n increases keeping other parameters fixed. It is noticed that MSEs of estimators of all unknown parameters decrease as τ_2 increase keeping other parameters fixed in the case of order restricted inference also. They also decrease as n increases. It is also observed that MSEs of estimators of all unknown parameters are smaller in case of order restricted inference than those in the unrestricted case.

6.4.2 Data Analysis

Table 6.15: Data for illustrative example.

Stress Level	Data							
1	0.1526	0.3381	0.3891	0.3936	0.4684	0.4716	0.4783	0.5575
	0.5685							
2	0.6009	0.6144	0.6276	0.6563	0.6566	0.6591	0.6629	0.6693
	0.6776	0.6948	0.6958	0.7089	0.7097	0.7113	0.7385	0.7679

Table 6.16: CRIs for unknown parameters for data in Table 6.15 under unrestricted priors.

		β		λ_1		λ_2	
Level	Type of CRI	LL	UL	LL	UL	LL	UL
90%	Symm. CRI	1.270	3.717	0.344	1.997	1.580	3.926
	HPD CRI	1.095	3.474	0.228	1.643	1.448	3.725
95%	Symm. CRI	1.120	4.038	0.290	2.382	1.437	4.249
	HPD CRI	1.053	3.891	0.195	2.023	1.328	4.053
99%	Symm. CRI	0.842	4.665	0.208	3.519	1.156	5.034
	HPD CRI	0.816	4.568	0.173	3.007	1.048	4.739

Table 6.17: CRIs for unknown parameters for data in Table 6.15 under order restricted priors.

		β		λ_1		λ_2	
Level	Type of CRI	LL	UL	LL	UL	LL	UL
90%	Symm. CRI	1.4974	3.2713	0.4513	1.5690	1.5643	3.4564
	HPD CRI	1.6980	3.2713	0.4037	1.4929	1.5643	3.4564
95%	Symm. CRI	1.3286	3.2713	0.3842	1.8634	1.4331	3.8452
	HPD CRI	1.4974	3.2713	0.3447	1.6308	1.3516	3.6096
99%	Symm. CRI	1.0569	3.2713	0.2895	1.8881	1.1899	4.3495
	HPD CRI	1.1487	3.2713	0.3213	1.9077	1.0110	4.0856

In this section we present a data analysis to illustrate the procedures described in Section 6.3. The data given in Table 6.15 is considered for this purpose. This data is artificially generated from KHM with $\beta = 2$, $\lambda_1 = 0.833$, $\lambda_2 = 2.222$, $\tau_1 = 0.6$, $\tau_2 = 0.8$, and $n = 40$. The priors assumptions are same as in Section 6.4.1. The estimates of β, λ_1, and λ_2 are 2.35, 0.93, and 2.61, respectively, in case of unrestricted inference, whereas in case of order restricted inference they are 2.49, 1.01, and 2.50, respectively. Symmetric and HPD CRI of unknown parameters are reported in Table 6.16 for unrestricted priors and in Table 6.17. Plot of marginal posterior

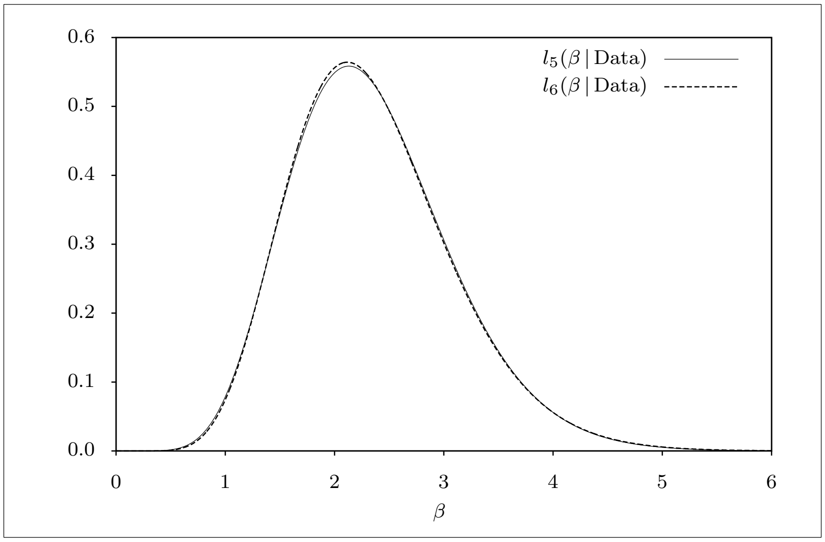


Figure 6.1: Gamma Approximation to $l_5(\beta \mid \mathrm{Data})$.

density function of β and its gamma approximation is provided in Figure 6.1 which depicts that the approximation is quite nice at least for this data set.

6.5 Posterior Analysis under Other Censoring Schemes

Type-II Censoring Scheme

Based on the observed sample, the likelihood function is given in (6.8), where $\tau^* = t_{r:n}$, in Case (a), $n_1^* = 0$, $n_2^* = r$, in Case (b), $n_1^* = n_1$, $n_2^* = r - n_1$ and in Case (c), $n_1^* = r$, $n_2^* = 0$. $D_1(\beta)$ and $D_2(\beta)$ have the same expression as given in case of Type-I censoring.

Type-I Hybrid Censoring Scheme

Based on the data from Type-I HCS, the likelihood function is same as (6.8), where in Case (a), $n_1^* = 0$, $n_2^* = r$, in Case (b), $n_1^* = n_1$, $n_2^* = r - n_1$, in Case (c), $n_1^* = r$, $n_2^* = 0$, in Case (d), $n_1^* = 0$, $n_2^* = n_2$, in Case (e), $n_1^* = n_1$, $n_2^* = n_2$, and in Case (f),

$n_1^* = n_1$, $n_2^* = 0$. Also in the Cases (a)-(b), $\tau^* = t_{r:n}$, where for the rest of the cases $\tau^* = \tau_2$. $D_1(\beta)$ and $D_2(\beta)$ have the same expression as given in case of Type-I censoring.

Type-II Hybrid Censoring Scheme

Based on the observed sample from Type-II HCS, the likelihood function is given in (6.8), where in Case (a), $n_1^* = 0$, $n_2^* = r$, for Case (b), $n_1^* = n_1$, $n_2^* = r - n_1$, in Case (c), $n_1^* = 0$, $n_2^* = n_2$, for Case (d), $n_1^* = n_1$, $n_2^* = n_2$ and for Case (e), $n_1^* = n_1$, $n_2^* = 0$. $\tau^* = t_{r:n}$ for Cases (a) and (b), whereas for the rest of the cases $\tau^* = \tau_2$. $D_1(\beta)$ and $D_2(\beta)$ have the same expression as given in case of Type-I censoring.

Progressive Type-II Censoring Scheme

With the observed Progressive Type-II censoring data, the likelihood function is given by (6.8), where for Case (a), $n_1^* = 0$, $n_2^* = m$, for Case (b), $n_1^* = n_1$, $n_2^* = m - n_1$ and for Case (c) $n_1^* = m$, $n_2^* = 0$. For all the cases $\tau^* = t_{m:n}$, $D_1(\beta) = \sum_{k=1}^{n_1^*}(R_k+1)t_{k:n}^{\beta} + (n - n_1^* - \sum_{k=1}^{n^*} R_k)\tau_1^{\beta}$ and $D_2(\beta) = \sum_{k=n_1^*+1}^{m}(R_k+1)(t_{k:n}^{\beta} - \tau_1^{\beta})$.

In all the above cases, likelihood function are in the same form as Type-I censoring scheme and hence, the posterior density will also be in the same form as given in (6.9). In all these cases we will be able to compute the BE and construct the associated CRI for some function of unknown parameters exactly along the same line.

6.6 Conclusion

A simple SSLT has been considered under the Bayesian framework. It has been assumed that the lifetimes at each stress level have a Weibull distribution with common shape parameter and different scale parameters. Analysis has been performed under KHM assumption. We have discussed both unrestricted and order restricted inference of the unknown parameters. It is noticed that in most of the cases BE of some function of unknown parameters cannot be obtained in close form, when they do exist. We have proposed algorithms based on the importance sampling to compute BE and to construct associate CRI of some parametric function. An extensive simulation has been also performed to judge the performance of the algorithms proposed. It is noticed that the proposed methods are working quite well for large values of n. For small values of n, MSEs of unknown parameters are quite large. Also the CPs of different CRIs are quite satisfactory for all the parameters under unrestricted inference, when 'almost' non-informative priors are used. It is also noticed that CPs of symmetric and HPD CRI of β are quite smaller than its nominal level for some choices of τ_1 and τ_2 under the order restricted priors. However, CPs maintain its nominal level for other parameters under the same priors. It is also noticed that MSEs of BE of unknown parameters are less in case of order restricted inference than those of unrestricted case. However, the results are quite prior dependent. The choice of proper priors is an important issue, which has not been pursued here and more work is needed in that direction.

6.A Appendix

6.A.1 Integrability Conditions for Unrestricted Case

Note that $A_1(\beta) > 0$ and $A_2(\beta) > 0$ for all $\beta > 0$. Also $n_1^* + a_1 > 0$ and $n_2^* + a_2 > 0$. Now

$$\int_0^\infty \int_0^\infty \int_0^\infty l_2(\beta,\, \lambda_1,\, \lambda_2 \,|\, \text{Data}) d\lambda_1 d\lambda_2 d\beta \propto \int_0^\infty l_5(\beta \,|\, \text{Data}) d\beta,$$

where $l_5(\beta \,|\, \text{Data})$ is given in (6.14). Let us define

$$\tau_1^* = \begin{cases} \tau_1 & \text{if } n - n_1^* > 0 \\ t_{n:n} & \text{if } n - n_1^* = 0, \end{cases} \qquad \tau_2^* = \begin{cases} \tau_2 & \text{if } n - n^* > 0 \\ t_{n:n} & \text{if } n - n^* = 0. \end{cases}$$

Case I : $0 < \tau_1^* < \tau_2^* < 1$

In this case, $0 < t_{i:n} < 1$ for all $i = 1, 2, \ldots, n_1^* + n_2^*$ and hence, $A_1(\beta) \to b_1$, $A_2(\beta) \to b_2$ as $\beta \to \infty$. For some positive constants c_2 and c_3,

$$c_2 \int_0^\infty \beta^{n^*+a_3-1} e^{-(b_3-c_1)\beta} d\beta \le \int_0^\infty l_5(\beta \,|\, \text{Data}) d\beta \le c_3 \int_0^\infty \beta^{n^*+a_3-1} e^{-(b_3-c_1)\beta} d\beta.$$

Clearly $l_5(\beta \,|\, \text{Data})$ is integrable if $n^* + a_3 > 0$ and $b_3 - c_1 > 0$. As $c_1 = \displaystyle\sum_{i=1}^{n_1^*+n_2^*} \ln t_{i:n} < 0$, $l_2(\beta, \lambda_1, \lambda_2 \,|\, \text{Data})$ is integrable if $n_1^* + a_1 > 0$, $n_2^* + a_2 > 0$, $n^* + a_3 > 0$, and $b_3 > 0$.

Case II : $0 < \tau_1^* < 1 < \tau_2^*$

In this case, $0 < t_{i:n} < 1$ for $i = 1, 2, \ldots, n_1^*$ and hence, $A_1(\beta) \to b_1$ as $\beta \to \infty$. As $\beta \to \infty$, $\dfrac{A_2(\beta)}{\tau_2^{*\beta}} \to (n - n^*)$ if $n - n^* > 0$. If $n - n^* = 0$, $\dfrac{A_2(\beta)}{\tau_2^{*\beta}} \to 1$ as $\beta \to \infty$. Hence, for some positive constants c_2 and c_3,

$$\begin{aligned} c_2 \int_0^\infty \beta^{n^*+a_3-1} e^{-(b_3-c_1+(n_2^*+a_2)\ln \tau_2^*)\beta} d\beta &\le \int_0^\infty l_5(\beta \,|\, \text{Data}) d\beta \\ &\le c_3 \int_0^\infty \beta^{n^*+a_3-1} e^{-(b_3-c_1+(n_2^*+a_2)\ln \tau_2^*)\beta} d\beta. \end{aligned}$$

Clearly $l_5(\beta \,|\, \text{Data})$ is integrable if $n^* + a_3 > 0$ and $b_3 - c_1 + (n_2^* + a_2) \ln \tau_2^* > 0$. Now

$$b_3 - c_1 + (n_2^* + a_2) \ln \tau_2^* = b_3 - \sum_{i=1}^{n_1^*} \ln t_{i:n} + \sum_{i=n_1^*+1}^{n^*} (\ln \tau_2^* - \ln t_{i:n}) + a_2 \ln \tau_2^*.$$

As $0 < t_{i:n} < 1$ for $i = 1, 2, \ldots, n_1^*$ and $t_{i:n} \le \tau_2^*$ for $i = n_1^* + 1, n_1^* + 2, \ldots, n^*$, $\displaystyle\sum_{i=1}^{n_1^*} \ln t_{i:n} < 0$ and $\displaystyle\sum_{i=n_1^*+1}^{n^*} (\ln \tau^* - \ln t_{i:n}) > 0$. Therefore $l_2(\beta, \lambda_1, \lambda_2 \,|\, \text{Data})$ is integrable if $n_1^* + a_1 > 0$, $n_2^* + a_2 > 0$, $n^* + a_3 > 0$, and $b_3 > 0$.

Case III : $1 < \tau_1^* < \tau_2^*$

In this case,

$$\frac{A_1(\beta)}{\tau_1^*} \to \begin{cases} n - n_1^* & \text{if } n - n_1^* > 0 \\ 1 & \text{if } n - n_1^* = 0, \end{cases} \qquad \frac{A_2(\beta)}{\tau_2^*} \to \begin{cases} n - n^* & \text{if } n - n^* > 0 \\ 1 & \text{if } n - n^* = 0, \end{cases}$$

as $\beta \to \infty$. Hence, for some positive constants c_2 and c_3,

$$c_2 \int_0^\infty \beta^{n^*+a_3-1} e^{-(b_3-c_1+(n_1^*+a_1)\ln\tau_1^*+(n_2^*+a_2)\ln\tau_2^*)\beta} d\beta \le \int_0^\infty l_5(\beta \,|\, \text{Data}) d\beta$$
$$\le c_3 \int_0^\infty \beta^{n^*+a_3-1} e^{-(b_3-c_1+(n_1^*+a_1)\ln\tau_1^*+(n_2^*+a_2)\ln\tau_2^*)\beta} d\beta.$$

Clearly $l_5(\beta \,|\, \text{Data})$ is integrable if $n^*+a_3>0$ and $b_3-c_1+(n_1^*+a_1)\ln\tau_1^*+(n_2^*+a_2)\ln\tau_2^*>0$. Now

$$\begin{aligned} &b_3-c_1+(n_1^*+a_1)\ln\tau_1^*+(n_2^*+a_2)\ln\tau_2^* \\ &= b_3+\sum_{i=1}^{n_1^*}(\ln\tau_1^*-\ln t_{i:n})+\sum_{i=n_1^*+1}^{n^*}(\ln\tau^*-\ln t_{i:n})+a_1\ln\tau_1^*+a_2\ln\tau_2^*. \end{aligned}$$

As $t_{i:n} \le \tau_1^*$ for $i=1, 2, \ldots, n_1^*$ and $t_{i:n} \le \tau_2^*$ for $i=n_1^*+1, n_1^*+2, \ldots, n^*$, $\sum_{i=1}^{n_1^*}(\ln\tau_1^*-\ln t_{i:n})>0$ and $\sum_{i=n_1^*+1}^{n^*}(\ln\tau^*-\ln t_{i:n})>0$. Therefore $l_2(\beta, \lambda_1, \lambda_2 \,|\, \text{Data})$ is integrable if $n_1^*+a_1>0$, $n_2^*+a_2>0$, $n^*+a_3>0$, and $b_3>0$.

Thus $l_2(\beta, \lambda_1, \lambda_2, \,|\, \text{Data})$ is integrable if proper priors are assumed on the unknown parameters for unrestricted inference case.

6.A.2 Integrability Conditions for Restricted Case

Note that $n^*+a_2>0$ and $\alpha D_1(\beta)+D_2(\beta)+b_2>0$ for all $\beta>0$ and $\alpha \in (0,1)$. Now

$$\int_0^\infty l_7(\alpha, \beta, \lambda_2 \,|\, \text{Data}) d\lambda_2 \propto \frac{\alpha^{n_1^*+a_4-1}(1-\alpha)^{b_4-1}\beta^{n^*+a_3-1}e^{-(b_3-c_1)\beta}}{\{\alpha D_1(\beta)+D_2(\beta)+b_2\}}. \tag{6.23}$$

Case I : $0<\tau_2^*<1$

For fixed $\alpha \in (0,1)$, $\alpha D_1(\beta)+D_2(\beta)+b_2 \to b_2$ as $\beta \to \infty$, and hence, right hand side of (6.23) is integrable with respect to $\alpha \in (0,1)$ and $\beta>0$ if $n^*+a_4>0$, $b_4>0$, $n^*+a_3>0$, and $b_3>0$. Therefore $l_7(\alpha, \beta, \lambda_2 \,|\, \text{Data})$ is integrable if proper priors are assumed on the unknown parameters.

Case II : $\tau_2^* \geq 1$

For fixed $\alpha \in (0,1)$, as $\beta \to \infty$,

$$\frac{\alpha D_1(\beta) + D_2(\beta) + b_2}{\tau_2^{*\beta}} \to \begin{cases} n - n^* & \text{if } n - n^* = 1 \\ 1 & \text{if } n - n^* = 0, \end{cases}$$

which is independent of $\alpha \in (0,1)$. Hence, in this case also, right hand side of (6.23) is integrable with respect to $\alpha \in (0,1)$ and $\beta > 0$ if $n^* + a_4 > 0$, $b_4 > 0$, $n^* + a_3 > 0$, and $b_3 > 0$. Therefore $l_7(\alpha,\, \beta,\, \lambda_2 \,|\, \text{Data})$ is integrable under the same condition as above. Thus $l_7(\alpha,\, \beta,\, \lambda_2 \,|\, \text{Data})$ is a proper PDF whenever proper priors are assumed on the unknown parameters in the case of order restricted inference.

Chapter 7

Conclusion and Future Work

7.1 Conclusion

In Chapter 2, we have considered the HCS-II, when lifetimes are assumed to have two-parameter exponential distribution. We have found the MLEs for both the parameters. We have considered different methods for construction of CI. We have seen that the approximate and BCa bootstrap methods for construction of CI are quite good. Also we approximate the distribution of the MLE of the scale parameter by gamma distribution and find the associated CI. It is noticed that this method of constructing CI is also performing quite well. Hence, we recommend to use the gamma approximation or BCa bootstrap method for constructing CI specially when n is large.

In Chapter 3, we have considered the Bayesian inference of the two-parameter exponential model when the data are hybrid or progressively censored. We have assumed a uniform prior on the location parameter and gamma prior on the scale parameter. The Bayes estimates may not be obtained explicitly in many cases, even when they exist. We have suggested to use the Monte Carlo sampling to compute simulation consistent Bayes estimators and also to construct the credible intervals. Monte Carlo simulation results suggest that the proposed Bayes estimators work quite well.

In Chapter 4, we have considered the Bayesian estimation in a simple SSLT imposing the order restriction on mean lifetimes under different censoring schemes where the data are coming from exponential CEM. We have taken mainly the square error loss function, though other loss functions can also be handled in a very similar way. We have seen that the BE of some parametric function under the square error loss function does not exist in close form in most of the cases. We have proposed to use the importance sampling to compute BE and CRIs. We have done a simulation study to judge the performance of the procedures described. We also considered two data sets to illustrate the estimation procedures. We have noticed that the performance of the CRIs for λ_1 and λ_2 are not at all satisfactory for small values of n. However, CRIs are quite well for moderate or large values of n, even if we do not use any prior information. HPD CRI works well for λ_1, where symmetric CRI works well for λ_2. Therefore we recommend to use HPD CRI for λ_1 and symmetric CRI for λ_2.

The two-parameter exponential distribution has been considered under a simple step-stress model in Chapter 5. The exact confidence limits of the scale parameters are difficult to obtain, due to the complicated nature of the model. We have proposed to use asymptotic and parametric bootstrap confidence intervals, and the performance of the later is better. We have further proposed Bayesian inference of the unknown parameters under fairly general prior assumptions, and we obtained the Bayes estimates and the associated credible intervals using importance sampling technique. The proposed Bayes estimates and the credible intervals perform quite well.

In Chapter 6, a simple SSLT has been considered under the Bayesian framework. It has been assumed that the lifetimes at each stress level have a Weibull distribution with common shape parameter and different scale parameters. Analysis has been performed under KHM assumption. We have discussed both unrestricted and order restricted inference of the unknown parameters. It has been noticed that in most of the cases BE of some function of unknown parameters cannot be obtained in close form as in Chapters 3 and 4. Algorithms based on the importance sampling have been proposed to compute BE and to construct associate CRI of some parametric function. An extensive simulation has been also performed to judge the performance of the algorithms proposed. The proposed methods are working quite well for large values of n. However, MSEs of unknown parameters are quite large for small values of n. Also the CPs of different CRIs are quite satisfactory for all the parameters under unrestricted and restricted inference, when 'almost' non-informative priors are used. It has been also noted that MSEs of BE of unknown parameters are less in case of order restricted inference than those of unrestricted case. However, the whole analysis is quite prior dependent.

7.2 Future Work

Open Problem 1. In Chapter 2, approximate confidence interval of the scale parameter is discussed assuming the stochastic monotonicity of the MLE for the scale parameter. Due to the complicated nature of the CDF of MLE for the scale parameter, we have verified the assumption numerically. Therefore the formal proof of the same remains an open problem.

Open Problem 2. In this dissertation, we have considered step-stress model, where a stress level is changed to the next stress level at a pre-fixed time. However, one may think of changing a stress level to the next stress level at a random time. See Xiong and Milliken [138], Xiong et al. [139], and Kundu and Balakrishnan [89] in this respect. In most of the cases the analysis is performed under the assumption of exponential distribution, and other distributions are not considered till now under this setup.

Open Problem 3. Several authors considered optimality of a SSLT in literature. In all these articles optimality is done under frequentist setup. They derived the observed fisher information matrix and A-optimality or D-optimally criteria are mainly used to get optimum step-stress plan. Bayesian analysis of SSLT opens the door for finding optimal SSLT under the same paradigm. Note that BE is found by minimizing Bayes risk and hence, one may think of minimizing Bayes risk with respect to stress changing time to find optimum plan.

Open Problem 4. Prior elicitation is becoming a popular topic among Bayesian. It will be a challenging task to find a subjective prior for step-stress life testing models.

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