Accelerated Life Tests

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Main Sections

1. Censoring
2. Accelerated Life Test
3. Step-stress Life Test
4. Review of Works
5. Multiple Failure Modes
6. Random Stress Changing Time
7. Order Restricted Inference
8. Optimal Design of SSLTs
9. Stage Life Tests

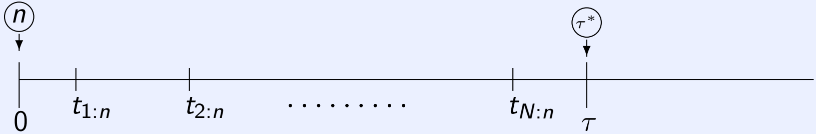
Censoring

Censoring

- Quite useful technique in reliability life testing.
- Possible termination of experiment before failing all the experimental units.
- Lower cost in terms of money and time than full experiment.
- Survival experimental units can be used for further experiments.

Type-I Censoring

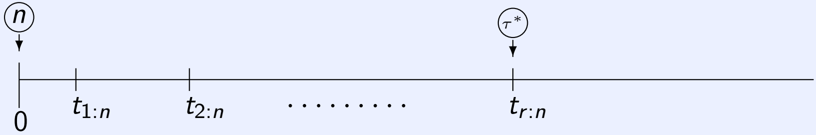
- n **:** Number of items put on the test.
- τ **:** Pre-fixed time.
- $\tau^* = \tau$ **:** Experiment termination time.



- Number of failures is a random variable.
- Advantage **:** Pre-fixed experiment termination time.
- Disadvantage **:** Very few failures, even no failure, before time τ.

Type-II Censoring

- n **:** Number of items put on the test.
- $r\,(\leq n)$ **:** Pre-fixed integer.
- $\tau^* = t_{r:n}$ **:** Experiment termination time.



- Duration of experiment is a random variable.
- Advantage **:** Pre-fixed number of failures.
- Disadvantage **:** Long experimental duration.

Other Censoring Schemes

- Hybrid censoring schemes: Mixture of Type-I and Type-II censoring schemes.
- Progressive censoring schemes: Allows the experimenter to remove units before its failure.

Accelerated Life Test

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- Put a sample of products on the test under one or more accelerated stresses to get early failures.
- Need to extrapolate to estimate the lifetime distribution under the normal condition.
- In practice, ALTs are performed in the presence of censoring.

ALT: Examples

- Food and Drug
  - Performance variables: pH, moisture loss or gain, microbial growth, color, specific chemical reaction.
  - Accelerating variables: Temperature, humidity, chemicals, pH, oxygen, solar radiation.
  - Societies: American Society of Test Methods, US Pharmacopoeia, Pharmaceutical Manufacturers Association.

ALT: Examples

- Nuclear Reactor
  - Performance variables: Strength, creep, creep-rupture.
  - Accelerating variables: Temperature, mechanical stress, contaminants, nuclear radiation.
  - Societies: Institute of Environmental Sciences, American Nuclear Society.

Stress Loading

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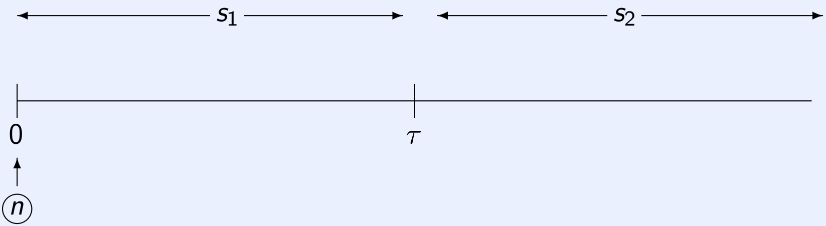
Step-stress Life Tests

Step-stress Life Tests

- A particular type of accelerated life test.
- Allows to change the stress levels during the life test.
- n : Number of items put on the test.
- s_1, s_2 : Stress levels (Simple SSLT).
- τ : Stress changing time (Pre-fixed).



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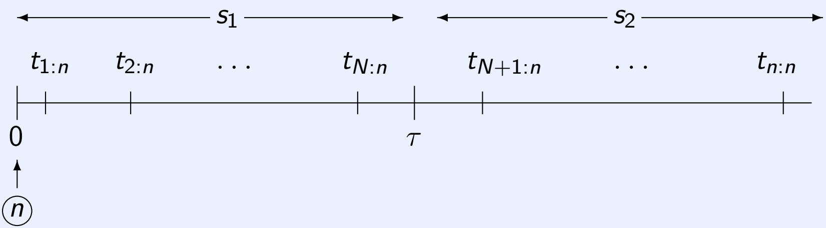
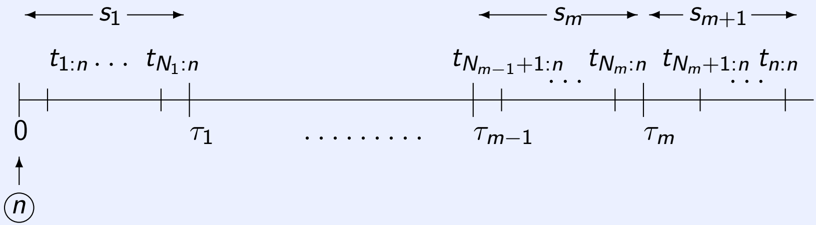


Step-stress Life Tests

- Generalization
  - n : No of items placed on the test.
  - $s_1, s_2, s_3, \ldots, s_{m+1}$: Stress levels.
  - $\tau_1 < \tau_2 < \ldots < \tau_m$: Stress changing times (Pre-fixed).



SUAV reliability data set[1]: An example of SSLT

- SUAV: Small unmanned aerial vehicle.
- Aircraft under the autonomous control with a dual propulsion system operating independently.

[1] Han, D. (2015) Estimation in step-stress life tests with complementary risks from the exponentiated exponential distribution under time constraint and its applications to UAV data, *Statistical Methodology*, 23:103–122.

SUAV reliability data set[1]: An example of SSLT

- SUAV: Small unmanned aerial vehicle.
- Aircraft under the autonomous control with a dual propulsion system operating independently.
- Simple step-stress experiment.
- Accelerating variables: Wind speed.
- Failure time: Both propulsions fail.
- $n = 23$, $\tau_1 = 15$ hours, $\tau_2 = 20$ hours.

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- Accelerating variables: Wind speed.
- Failure time: Both propulsions fail.
- $n = 23$, $\tau_1 = 15$ hours, $\tau_2 = 20$ hours.
- Failure times at first stress level: 2.365, 3.467, 5.386, 7.714, 9.578, 9.683, 11.416, 11.789, 12.039, 14.928, 14.938.
- Failure times at second stress level: 15.325, 15.781, 16.105, 16.362, 17.178, 17.366, 17.803, 19.578.

[1] Han, D. (2015) Estimation in step-stress life tests with complementary risks from the exponentiated exponential distribution under time constraint and its applications to UAV data, *Statistical Methodology*, 23:103–122.

Advantages

- By increasing the stress level, reasonable number of failure can be obtained.
- Experiment time is reduced.

Disadvantages

- Exact relationship between the stress level and lifetime of the product may be needed.
- Model must take into account the effect of stress accumulated.
- Model becomes more complicated.

Models

- $F_i(.)$: CDF of lifetime of an item under the stress level s_i, $i = 1, 2, \ldots, m$.
- $F(.)$ is the CDF of lifetime of an item under the step-stress pattern.
- Model needed to relate $F(\cdot)$ to $F_i(\cdot)$, $i = 1, 2, \ldots, m$.
- Popular models
  - Cumulative exposure model.
  - Tampered random variable model.
  - Tampered failure rate model.
  - Cumulative risk model.

Cumulative exposure model

- Possibly the most popular model.
- First proposed by Seydyakin (1966)[2] and later studied by Nelson(1980)[3].
- $F_i(\cdot)$ is the CDF of lifetime of an item under the stress level s_i, $i = 1, 2, \ldots, m+1$.
- $F(\cdot)$ is the CDF of lifetime of an item under the step-stress pattern.

[2] Seydyakin, N. M. (1966) On one physical principle in reliability theory, *Technical Cybernatics*, 3:80-87.

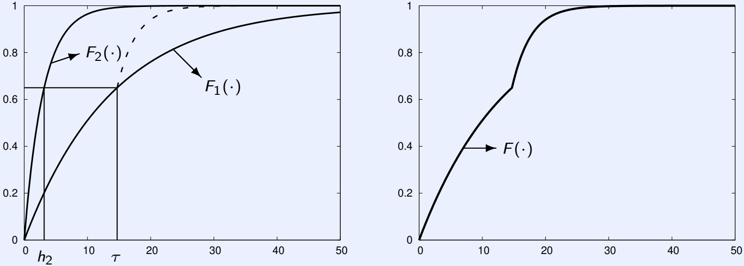
[3] Nelson (1980) Accelerated life testing: step-stress models and data analysis, *IEEE Transactions on Reliability*, 141:288-2838.

Cumulative exposure model

The CEM assumption is:

- If the stress level is fixed, the survivors will fail according to the distribution function of that stress level but starting at previous accumulated fraction failed.

Cumulative exposure model



(a) CDF under different stress level (b) CDF under CEM

Figure: Graphical representation of CEM

Here $F_1(\cdot)$ and $F_2(\cdot)$ are CDF of $Exp(14)$ and $Exp(1)$ respectively.

Cumulative exposure model

Under the assumptions of CEM the CDF of the lifetime is given by

$$F(t) = F_i(t - \tau_{i-1} + h_{i-1}) \quad \text{if } \tau_{i-1} \le t < \tau_i,\ i = 1, 2, \ldots, m+1,$$

where $\tau_0 = 0$, $\tau_{m+1} = \infty$, $h_0 = 0$ and h_i, $i = 1, 2, \ldots, m$, is the solution of

$$F_{i+1}(h_i) = F_i(\tau_i - \tau_{i-1} + h_{i-1}).$$

Tampered random variable model

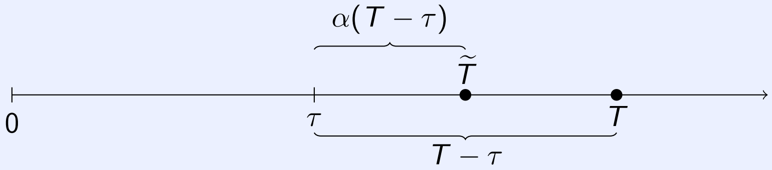
- Proposed by DeGroot and Goel (1979)[4] for simple SSLT.
- T: Lifetime of the unit under first stress level.
- The effect of change in the stress level at the time τ is equivalent to multiply the remaining lifetime by an unknown positive constant, say α.

$$\widetilde{T} = \begin{cases} T & \text{if } T \leq \tau \\ \tau + \alpha(T - \tau) & \text{if } T > \tau. \end{cases}$$

[4] DeGroot, M. H. and Goel, P. K. (1979) Bayesian estimation and optimal design in partially accelerated life testing, *Naval Research Logistics*, 26:223–235.

Tampered random variable model

$$\widetilde{T} = \begin{cases} T & \text{if } T \le \tau \\ \tau + \alpha(T - \tau) & \text{if } T > \tau. \end{cases}$$



Tampered random variable model

- Generalized by Sultana and Dewanji (2021)[5] for multiple step-stress.

$$\widetilde{T} = \begin{cases} T & \text{if } 0 \leq T \leq \tau_1^* \\ \tau_1^* + \alpha_1(T - \tau_1^*) & \text{if } \tau_1^* < T \leq \tau_2^* \\ \tau_2^* + \alpha_1\alpha_2(T - \tau_2^*) & \text{if } \tau_2^* < T \leq \tau_3^* \\ \vdots & \\ \tau_m^* + (T - \tau_m^*)\prod_{i=1}^{m}\alpha_i & \text{if } T > \tau_m^*, \end{cases}$$

where $\tau_1^* = \tau_1$ and $\tau_i^* = \tau_{i-1}^* + \frac{\tau_i - \tau_{i-1}}{\prod_{j=1}^{i-1}\alpha_j}$ for $i = 2, 3, \ldots, m$.

[5] Sultana, F and Dewanji, A. (2021) Tampered random variable modeling for multiple step-stress life test, *Communications in Statistics - Theory and Methods*, DOI: 10.1080/03610926.2021.2008440

Tampered failure rate model

- Proposed by Bhattacharyya and Soejoeti (1989)[6] for simple SSLT.
- $\lambda(t)$: Failure rate at the first stress level.
- The effect of change in the stress level is to multiply the failure rate of first stress level by an unknown positive constant, say α.

$$\widetilde{\lambda}(t) = \begin{cases} \lambda(t) & \text{if } t \le \tau \\ \alpha\lambda(t) & \text{if } t > \tau. \end{cases}$$

[6] Bhattacharyya, G. k. and Soejoeti, Z. (1989) "A tampered failure rate model for step-stress accelerated life test", *Communication in Statistics - Theory and Methods*, 18:1627–1643.

Tampered failure rate model

- Generalized by Madi (1993)[7] for multiple SSLT.

$$\widetilde{\lambda}(t) = \lambda(t) \prod_{j=0}^{i-1} \alpha_j \text{ for } \tau_{i-1} < t \leq \tau_i,\ i = 1, 2, \ldots, m+1,$$

where $\tau_0 = 0$ and $\tau_{m+1} = \infty$.

---

[7] Madi, M. T. (1993) "Multiple step-stress accelerated life test: the tampered failure rate model", *Communication in Statistics - Theory and Methods*, 22:2631-2639.

Cumulative risk model

- The hazard function of CEM, TRVM, and TFRM are not continuous.
- This is may not be realistic in many practical situations.
- For example, the experimental setup will take some time to reach target temperature when it is increased.
- To overcome it, Drop et al. (1996)[8] considered CRM.

[8] Van Dorp, J. R., Mazzuchi, T. A., Fornell, G. E. and Pollock, L. R. (1996). A Bayes approach to step-stress accelerated life testing. *IEEE Transactions on Reliability*, 45(3), 491-498.

Cumulative risk model

- Simple step-stress: s_1, s_2
- τ : Stress changing time.
- $h_1(\cdot)$, $h_2(\cdot)$: Hazard functions (continuous) under s_1 and s_2.
- δ : Latency period.
- The hazard function under simple step-stress is

$$\widetilde{h}(t) = \begin{cases} h_1(t) & \text{if } 0 < t \leq \tau \\ \rho(t) & \text{if } \tau < t \leq \tau + \delta \\ h_2(t) & \text{if } t > \tau + \delta, \end{cases}$$

where $\rho(\cdot)$ is a continuous increasing function satisfying $h_1(\tau) = \rho(\tau)$ and $h_2(\tau + \delta) = \rho(\tau + \delta)$.

Cumulative risk model

- $\rho(\cdot)$: Linear in t (This is popular choice).

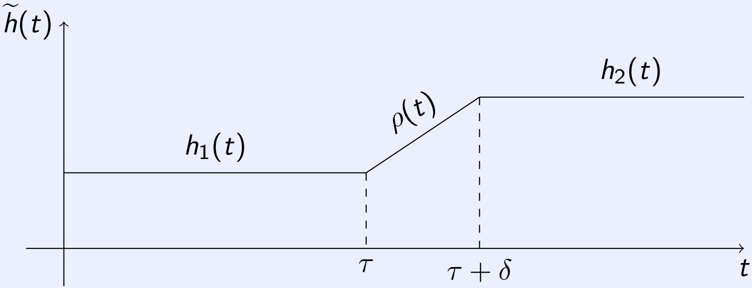
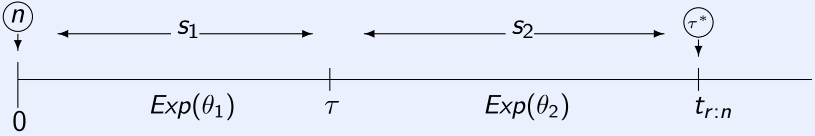


Figure: Illustration of CRM: $h_1(t) = \lambda_1$, $h_2(t) = \lambda_2$

Review of Works

Balakrishnan et al. (2007)[9].

- Simple step-stress life test.
- Type-II censoring.
- Exponentially distributed failure times.
- Cumulative exposure model.



[9] Balakrishnan, N., Kundu, D., Ng, H. K. T., and Kannan, N. Point and interval estimation for a simple step-stress model with Type-II censoring. *Journal of Quality Technology*, 9:35–47, 2007.

Balakrishnan et al. (2007)[9].

- Point and interval estimation is considered.
- MLEs of θ_1 and θ_2 exist if there is at least one failure in each stress level.
- The authors derived the exact conditional distributions of the MLEs of θ_1 and θ_2.
- The PDFs are quite complicated. However, PDFs can be expressed as generalized mixture of shifted Gamma PDFs.
- Proposed a method to construct confidence interval of model parameters based on conditional distribution.

[9] Balakrishnan, N., Kundu, D., Ng, H. K. T., and Kannan, N. Point and interval estimation for a simple step-stress model with Type-II censoring. *Journal of Quality Technology*, 9:35–47, 2007.

Balakrishnan et al. (2009)[10].

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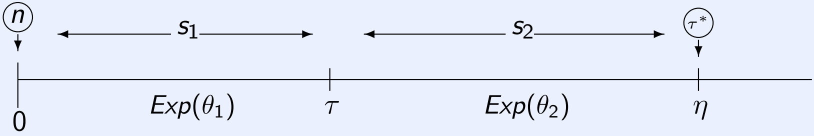

[10] Balakrishnan, N., Xie, Q., and Kundu, D. Exact inference for a simple step-stress model from the exponential distribution under time constrain. *Annals of the Institute of Statistical Mathematics*, 61:251–274, 2009.

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Other works on CEM

- Several authors then considered CEM under different lifetime distributions including
  - Two-parameter exponential distribution
  - Weibull distribution
  - Generalized exponential distribution
  - Gamma distribution
  - Log-normal distribution
  - Log-logistic distribution
  - Pareto distribution
  - Birnbaum-Saunders distribution
  - Geometric distribution

References are given at the end.

Literature on TRVM

- The TRVM is considered in the presence of different censoring schemes under different lifetime distributions including
  - Exponential distribution
  - Weibull distribution
  - Generalized exponential distribution
  - Generalized weibull distribution
  - Pareto distribution
  - Lomax distribution
  - Log-normal distribution

References are given at the end.

Literature on TFRM

- The inferential issues for TFRM is studied for Weibull distribution.

References are given at the end.

Literature on CRM

- The inferential issues for CRM is considered under the assumption of following hazard function:

$$h(t) = \begin{cases} \lambda_i & \text{if } \tau_{i-1} + \xi_{i-1} < t \le \tau_i, \\ & \quad i = 1, 2, \ldots, m+1 \\ \lambda_i + \frac{\lambda_{i+1} - \lambda_i}{\xi_i}(t - \tau_i) & \text{if } \tau_i < t \le \tau_i + \xi_i \\ & \quad i = 1, 2, \ldots, m, \end{cases}$$

with $\tau_0 = \xi_0 = 0$ and $\tau_{m+1} = \infty$.

Literature on CRM

- Classical Inference: Kannan et al. (2010)[11] .
- Bayesian Inference: Drop et al. (1996)[12] , and Drop and Mazzuchi (2004)[13] .

[11] Kannan, N., Kundu, D. and Balakrishnan, N. (2010). Survival models for step-stress experiments with lagged effects. In Advances in degradation modeling (pp. 355-369). Birkhäuser Boston.

[12] Van Dorp, J. R., Mazzuchi, T. A., Fornell, G. E. and Pollock, L. R. (1996). A Bayes approach to step-stress accelerated life testing. *IEEE Transactions on Reliability*, 45(3), 491-498.

[13] Van Dorp, J. R. and Mazzuchi, T. A. (2004). A general Bayes exponential inference model for accelerated life testing. *Journal of statistical planning and inference*, 119(1), 55-74.

Link Function

- k-step step-stress.
- Exponential distribution with mean θ_i at stress level s_i.
- CEM.
- MLEs of θ_i exists if at least one failure occur at stress level s_i.
- To overcome, link function is used.
- For example $\ln \theta_i = \alpha + \beta s_i$ (log-linear link function), $i = 1, 2, \ldots, k$.
- Need to estimate only α and β.

Bayesian Inference

- Likelihood based distribution and model.
- Priors distribution (Gamma).
- Posterior distribution.
- Most of the cases Bayes estimate do not exist in closed form.
- Importance sampling or MCMC are used.

Multiple Failure Modes

Competing Risks

- Exposure to more than one risks of failure.
- The risks compete among themselves to be the cause of failure.
- For a failed item, failure time and cause of failure are recorded.
- Needs to assess the effect of each cause in the presence of other causes.

Competing Risks

- Exposure to more than one risks of failure.
- The risks compete among themselves to be the cause of failure.
- For a failed item, failure time and cause of failure are recorded.
- Needs to assess the effect of each cause in the presence of other causes.
- Examples:
    - Causes of failure of an industrial heaters may be classified as "open" and "short".
    - A circuit board fails if any of its joints fail.
    - A ball bearing fails due to poor lubrication or material defects.
    - One individual may die due to heart attack or kidney failure or cancer.

Models for Competing Risks

- The latent failure time model of Cox (1959)[14]
- The cause specific hazard function approach of Prentice et al. (1978)[15] .

[14] Cox, D. R. (1959) "The analysis of exponentially distributed lifetimes with two types of failures", *Journal of the Royal Statistical Society*, 21:411-421.

[15] Prentice, R. L., Kalbfleisch, J. D., Peterson Jr, A. V., Flournoy, N., Farewell, V. T., and Breslow, N. E. (1978). The analysis of failure times in the presence of competing risks, *Biometrics*, 541-554.

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[17] Prentice, R. L., Kalbfleisch, J. D., Peterson Jr, A. V., Flournoy, N., Farewell, V. T., and Breslow, N. E. (1978). The analysis of failure times in the presence of competing risks, *Biometrics*, 541-554.

Cox's Latent Failure Time Model

- k competing causes.
- T_i is the (hypothetical) lifetime in the presence of only i-th causes, $i = 1, 2, \ldots, k$.
- The observed lifetime is $T = \min\{T_1, T_2, \ldots, T_k\}$.
- F_i and f_i are CDF and PDF of T_i.
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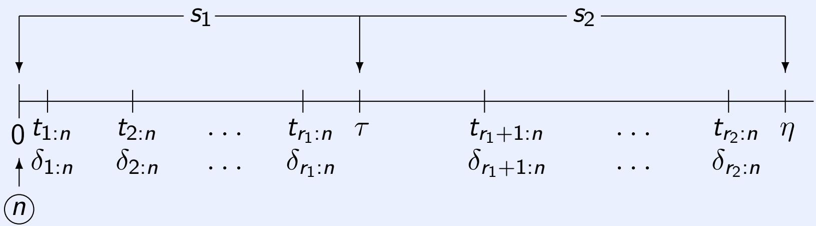
- F_i and f_i are CDF and PDF of T_i.
- In most of cases, T_i's are assumed to be independent.
- A failed observation is of the from (t, δ). It's likelihood contribution is

$$f_\delta(t) \prod_{i=1,\, i\neq\delta}^{k} (1 - F_i(t)).$$

- Likelihood contribution of an item censored at t is

$$\prod_{i=1}^{k}(1 - F_i(t)).$$

Form of CR-SSLT Data



CEM and Latent Failure Time Model

- 2 competing risks.
- 2 stress levels.

	Risk I	Risk II
Stress I	F_{11}	F_{21}
Stress II	F_{21}	F_{22}

- The CDFs under CEM are

$$\text{Risk I: } F_1(t) = \begin{cases} F_{11}(t) & \text{if } 0 < t \leq \tau \\ F_{12}(t - a_1) & \text{if } t > \tau \end{cases}$$

$$\text{Risk II: } F_2(t) = \begin{cases} F_{21}(t) & \text{if } 0 < t \leq \tau \\ F_{22}(t - a_2) & \text{if } t > \tau \end{cases}$$

- Based on the previous CDFs, likelihood function can be written.

CEM and Latent Failure Time Model

- Balakrishnan and Han (2008)[18] assumed exponential distributions and Type-II censoring.
- Han and Balakrishnan (2010)[19] assumed exponential distributions and Type-I censoring.

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[18] Balakrishnan, N. and Han, D. (2008), Exact inference for a simple step-stress model with competing risks for failure from exponential distribution under Type-II censoring. *Journal of Statistical Planning and Inference*, 138(12), 4172-4186.

[19] Han, D. and Balakrishnan, N. (2010), Inference for a simple step-stress model with competing risks for failure from the exponential distribution under time constraint. *Computational Statistics & Data Analysis*, 54(9), 2066-2081.

TFRM and Latent Failure Time Model

- 2 competing risks and 2 stress levels.
- h_1 and h_2 are hazard function of T_1 and T_2, respectively, under initial stress level.
- The hazard functions of T_1 and T_2 under TFRM are

$$\text{Risk I: } \widetilde{h}_1(t) = \begin{cases} h_1(t) & \text{if } 0 < t \leq \tau \\ \alpha_1 h_1(t) & \text{if } t > \tau \end{cases}$$

$$\text{Risk II: } \widetilde{h}_2(t) = \begin{cases} h_2(t) & \text{if } 0 < t \leq \tau \\ \alpha_2 h_2(t) & \text{if } t > \tau \end{cases}$$

- Now, likelihood function can be written and MLEs can be found.

TFRM and Latent Failure Time Model

- Assuming baseline hazard is Weibull, inferential issues are addressed by Zhang et al. (2016)[20] and Liu and Shi (2017)[21] for multiple SSLT (All α_i's are assumed same).

[20] Zhang, C., Shi, Y. and Wu, M. (2016). Statistical inference for competing risks model in step-stress partially accelerated life tests with progressively Type-I hybrid censored Weibull life data. *Journal of Computational and Applied Mathematics*, 297, 65-74.

[21] Liu, F. and Shi, Y. (2017). Inference for a simple step-stress model with progressively censored competing risks data from Weibull distribution. *Communications in Statistics-Theory and Methods*, 46(14), 7238-7255.

Complementary Risks

- Recall the example of SUAV reliability data.
- Failure time: Both propulsions fail.
- Cause: Last failed propulsion.
- $T = \max\{T_1, T_2\}$.
- This model is known as complementary risks in literature.
- Introduced by Basu and Ghosh (1980)[22] .
- Han (2015)[23] .
  - Two complementary risks
  - Simple SSLT
  - CEM
  - Likelihood inference

[22] Basu, A. P. and Ghosh, J. K. (1980). Identifiability of distributions under competing risks and complementary risks model. *Communications in Statistics-Theory and Methods*, 9(14), 1515-1525.

[23] Han, D. (2015). Estimation in step-stress life tests with complementary risks from the exponentiated exponential distribution under time constraint and its applications to UAV data. Statistical Methodology, 23, 103-122.

Random Stress Changing Time

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- A drawback: The model parameters are estimable if there is at least one failure at each stress level.

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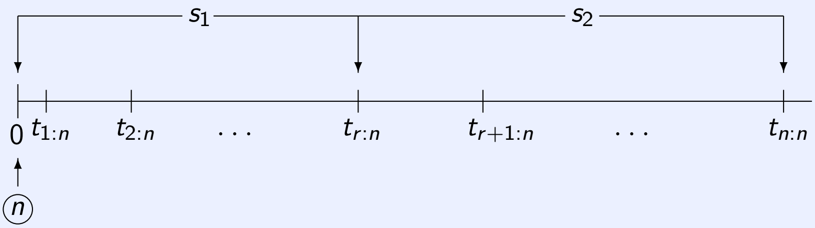
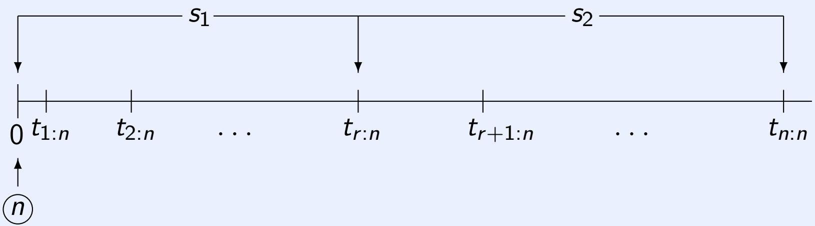


Random Stress Changing Time

- A drawback: The model parameters are estimable if there is at least one failure at each stress level.
- A solution: Change the stress level with pre-specified number of failures.

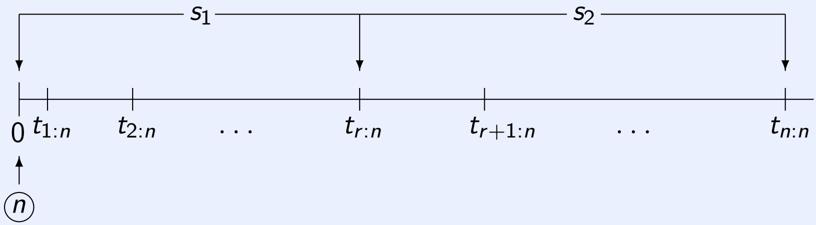


- Adjustment need to be done to accommodate random stress changing time.

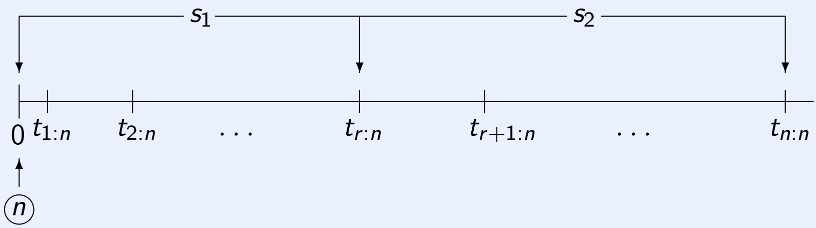
Likelihood Function under CEM

- Distribution: $Exp(\theta_1)$ and $Exp(\theta_2)$ at s_1 and s_2, respectively.
- If the stress changing time, τ, is pre-fixed, the CDF of the lifetime is given by

$$F(t) = \begin{cases} \frac{1}{\theta_1} e^{-\frac{t}{\theta_1}} & \text{if } 0 < t \leq \tau \\ \frac{1}{\theta_2} e^{-\frac{t-\tau}{\theta_2} - \frac{\tau}{\theta_1}} & \text{if } t > \tau. \end{cases}$$



Likelihood Function under CEM



- Distribution of $T_{1:n}, \ldots, T_{r:n}$ is same as that of first r order statistics of a sample of size n from an $Exp(\theta_1)$ distribution.
- Conditional distribution of $T_{r+1:n}, \ldots, T_{n:n}$ given $T_{1:n}, \ldots, T_{r:n}$ is same as that of order statistics of a sample of size $n-r$ from an $Exp(t_{r:n}, \theta_2)$ with PDF

$$f(t) = \frac{1}{\theta_2} e^{-\frac{t - t_{r:n}}{\theta_2}} \quad \text{if } t > t_{r:n}.$$

- Now, one can write likelihood function of θ_1 and θ_2.

Likelihood Function under CEM

- Xiong and Milliken (1999)[24] .
  - Scale family of distributions.
  - CEM
  - Complete data
- Kundu and Balakrishnan (2009)[25] .
  - Exponential distribution
  - CEM
  - Type-II censoring
  - Exact Inference

[24] Xiong, C. and Milliken, G. A. (1999). Step-stress life-testing with random stress-change times for exponential data. *IEEE Transactions on Reliability*, 48(2), 141-148.

[25] Kundu, D. and Balakrishnan, N. (2009). Point and interval estimation for a simple step-stress model with random stress-change time. *Journal of Probability and Statistical Science*, 7, 113-126.

Likelihood Function under CEM

- Zhang and Shi (2016)[26] .
  - Weibull distribution
  - TFRM
  - Adaptive progressively hybrid censoring
  - Multiple step-stress
  - Log-linear link function

[26] Zhang, C. and Shi, Y. (2016). Estimation of the extended Weibull parameters and acceleration factors in the step-stress accelerated life tests under an adaptive progressively hybrid censoring data. *Journal of Statistical Computation and Simulation*, 86(16), 3303-3314.

Order Restricted Inference

Order Restricted Inference

- To observed quick failures.
- Increase the stress level.
- Plausible to assume that mean lifetime decreases as stress level increases.
- Order restriction is very natural.

Exponential Lifetimes

- Simple step-stress
- Lifetime distributions: $Exp(\theta_1)$ and $Exp(\theta_2)$.
- Means of lifetime: θ_1 and θ_2.
- Order restriction: $\theta_1 \geq \theta_2$.
- This can be easily extended for multiple SSLT.

Statistical Inference

- Balakrishnan et al. (2009)[27] .
  - Multiple step-stress.
  - Exponential distribution.
  - CEM
  - Optimize likelihood function under order restriction using isotonic regression[28] .

[27] Balakrishnan, N., Beutner, E. and Kateri, M. (2009). Order restricted inference for exponential step-stress models. *IEEE Transactions on Reliability*, 58(1), 132-142.

[28] Barlow,R. E., Bartholomew, D. J., Bremner, J. M. and Brunk, H. D. (1972). Statistical Inference under Order Restrictions, *New York:John Wiley & Sons*.

Statistical Inference

- Implementation of isotonic regression is not very easy.
- An alternative: Reparametrization.
- For simple SSLT ($\theta_1 \geq \theta_2$):

  $(\theta_1, \theta_2) \to (\theta_1, \beta)$ by $\theta_2 = \beta\theta_1$, where $\beta \in (0, 1]$.
- Likelihood of θ_1 and β can be obtained and optimized over $\theta_1 > 0$ and $\beta \in (0, 1]$.

Statistical Inference

- For multiple SSLT ($\theta_1 < \theta_2 < \ldots < \theta_m$):
  $(\theta_1, \theta_2, \ldots, \theta_m) \rightarrow (\theta_1, \beta_1, \beta_2, \ldots, \beta_{m-1})$ by

$$\theta_2 = \beta_1 \theta_1$$
$$\theta_3 = \beta_2 \theta_2 = \beta_1 \beta_2 \theta_1$$
$$\vdots$$
$$\theta_m = \beta_{m-1} \theta_{m-1} = \beta_1 \beta_2 \ldots \beta_{m-1} \theta_1.$$

- Several authors used reparametrization and performed classical and Bayesian analysis under different censoring schemes and models.

References are given at the end.

Bayesian Inference

- Order restriction can be imposed using prior distribution.
- Prior distribution with ordering on parameters.
- Alternative: Use the reparametrization $\theta_2 = \beta\theta_1$.
    - Prior with positive support on θ_1 (For example, Gamma).
    - Prior with support on (0, 1) for β (For example, Beta).
    - Independent priors.

Optimal Design of SSLTs

Optimal Design of SSLTs

- Choice of stress changing times that provides "precise" estimates.
- Stress changing times: $0 < \tau_1 < \tau_2 < \ldots < \tau_m$.
- Choose a meaningful criterion and optimize it with respect to stress changing times, assuming other relevant quantities held fixed.

Some Popular Criteria

- A-optimal criterion
  - Trace of Fisher information matrix.
  - Minimized over $0 < \tau_1 < \tau_2 < \ldots < \tau_m$.

References are given at the end.

Some Popular Criteria

- A-optimal criterion
  - Trace of Fisher information matrix.
  - Minimized over $0 < \tau_1 < \tau_2 < \ldots < \tau_m$.
- C-optimal criterion
  - $AVar\left(\ln \widehat{\theta}_0\right)$, where θ_0 is the mean time to failure at use stress.
  - Minimized over $0 < \tau_1 < \tau_2 < \ldots < \tau_m$.

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- D-optimal criterion
  - Determinant of Fisher information matrix.
  - Maximized over $0 < \tau_1 < \tau_2 < \ldots < \tau_m$.

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Some Popular Criteria

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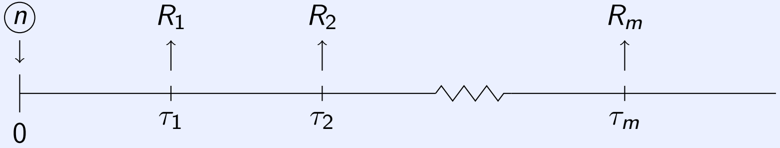
- D-optimal criterion
  - Determinant of Fisher information matrix.
  - Maximized over $0 < \tau_1 < \tau_2 < \ldots < \tau_m$.
- Several articles on optimal design with different criterion, different distributions, different models, different censoring schemes.

References are given at the end.

Stage Life Tests

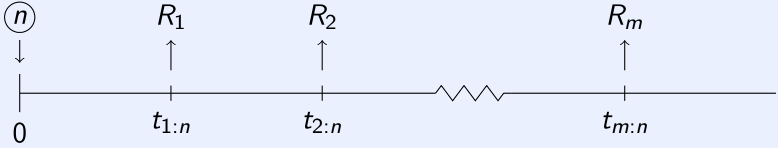
Progressive Censoring (Type-I)

- n : Number of item put on test
- $0 < \tau_1 < \tau_2 < \ldots < \tau_m$: Pre-fixed times.
- $R_1, R_2, \ldots, R_m$: Pre-fixed non-negative integers.
- $\sum_{i=1}^{m} R_i < n$.
- Removed items are utilized for other tests.



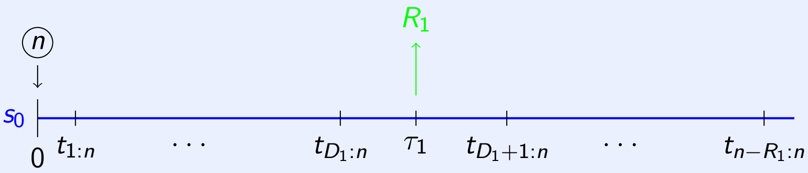
Progressive Censoring (Type-II)

- n : Number of item put on test.
- $R_1, R_2, \ldots, R_m$: Pre-fixed non-negative integers.
- $\sum_{i=1}^{m} R_i + m = n$.



Stage Life Tests (Type-I)

- Take $m = 1$.
- τ_1 : Pre-fixed time.
- R_1 : Pre-fixed non-negative integer.



Stage Life Tests (Type-I)

- Take $m = 1$.
- τ_1 : Pre-fixed time.
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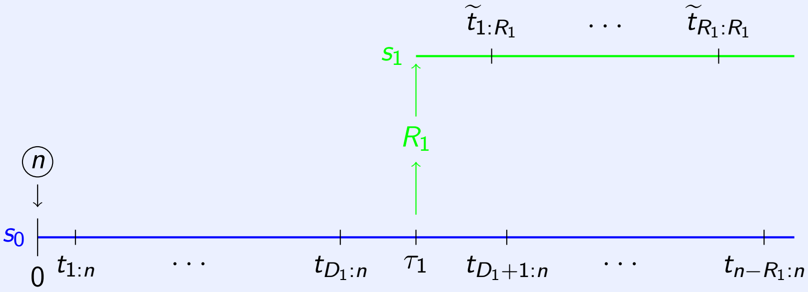

Works

- Laumen and Cramer (2019)[29] .
  - $m = 1$.
  - Progressive Type-I censoring.
  - Exponential Distribution.
  - CEM.
  - Distributional results and classical inference.
- Laumen and Cramer (2021)[30] .
  - Extended the previous work by taking $m = k$.
- Laumen and Cramer (2022)[31] .
  - Progressive Type-II censoring.

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[31] Laumen, B. and Cramer, E. (2022). Stage life testing with random stage changing times. *Communications in Statistics-Theory and Methods*, 51(12), 3934-3959.

Works

- Samanta et al. (2022)[32] .
  - Progressive Type-I censoring.
  - General value of m.
  - Weibull distribution.
  - TFRM.
  - Order restricted classical inference.
  - Optimal design: Optimum values of $R_1, R_2, \ldots, R_m$.

[32] Samanta, D., Mondal, S. and Kundu, D. (2022). Optimal plan for ordered step-stress stage life testing. *Statistics*, 1-26, DOI: 10.1080/02331888.2022.2152453.

Thank You

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