On Classical and Bayesian Order Restricted Inference for Multiple Exponential Step Stress Model

Ayon Ganguly
Department of Mathematics
Indian Institute of Technology Guwahati, India
aganguly@iitg.ernet.in

July 04, 2019

Paper

The talk is based on the following paper:

- D. Samanta, A. Ganguly, A. Gupta, and D. Kundu (2019), On Classical and Bayesian Order Restricted Inference for Multiple Exponential Step Stress Model, *Statistics*, 53:177–195.

Main Sections

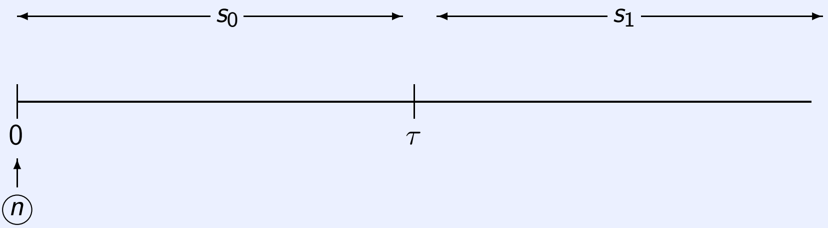
1. Step-stress Life Test
2. Models
3. Our Model
4. Likelihood Inference

Accelerated Life Tests

- Useful experimental technique to obtain data on the lifetime distribution of highly reliable products.
- Put a sample of products on the test under one or more accelerated stresses to get early failures.
- Need to extrapolate to estimate the lifetime distribution under the normal condition.
- In practice, ALTs are performed in the presence of censoring.

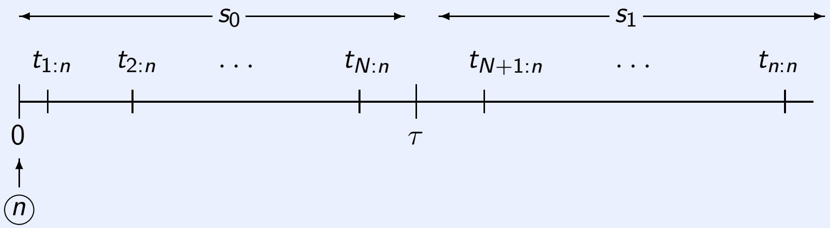
Step-stress Life Tests

- A particular type of accelerated life test.
- Allows to change the stress levels during the life test.
- n : Number of items put on the test.
- s_0, s_1 : Stress levels (Simple SSLT).
- τ : Stress changing time (Pre-fixed).



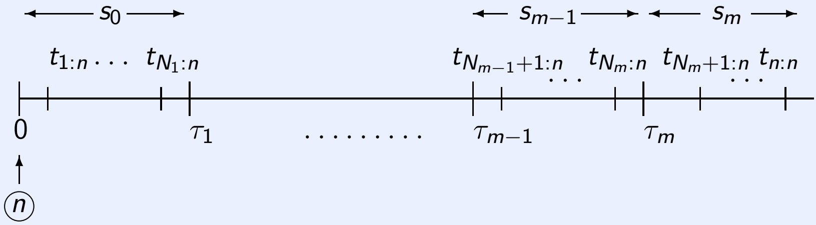
Step-stress Life Tests

- A particular type of accelerated life test.
- Allows to change the stress levels during the life test.
- n : Number of items put on the test.
- s_0, s_1 : Stress levels (Simple SSLT).
- τ : Stress changing time (Pre-fixed).



Step-stress Life Tests

- Generalization
  - n : No of items placed on the test.
  - $s_0, s_2, s_3, \ldots, s_m$: Stress levels.
  - $\tau_1 < \tau_2 < \ldots < \tau_m$: Stress changing times (Pre-fixed).



Models

- $F_i(.)$: CDF of lifetime of an item under the stress level s_i, $i = 0, 1, 2, \ldots, m$.
- $F(.)$ is the CDF of lifetime of an item under the step-stress pattern.
- Model needed to relate $F(\cdot)$ to $F_i(\cdot)$, $i = 0, 1, 2, \ldots, m$.
- Popular models
  - Tampered random variable model.
  - Tampered failure rate model.
  - Khamis-Higgins model.
  - Cumulative exposure model.

Models

- $F_i(.)$: CDF of lifetime of an item under the stress level s_i, $i = 0, 1, 2, \ldots, m$.
- $F(.)$ is the CDF of lifetime of an item under the step-stress pattern.
- Model needed to relate $F(\cdot)$ to $F_i(\cdot)$, $i = 0, 1, 2, \ldots, m$.
- Popular models
    - Tampered random variable model.
    - Tampered failure rate model.
    - Khamis-Higgins model.
    - **Cumulative exposure model**.

CEM

- Possibly the most popular model.
- First proposed by Seydyakin (1966)[1] and later studied by Nelson (1980)[2] .
- $F_i(\cdot)$ is the CDF of lifetime of an item under the stress level s_i, $i = 1, 2, \ldots, m + 1$.
- $F(\cdot)$ is the CDF of lifetime of an item under the step-stress pattern.

[1] Seydyakin, N. M. (1966) On one physical principle in reliability theory, *Technical Cybernatics*, 3:80-87.

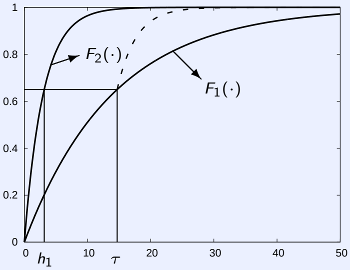
[2] Nelson (1980) Accelerated life testing: step-stress models and data analysis, *IEEE Transactions on Reliability*, 141:288-2838.

CEM

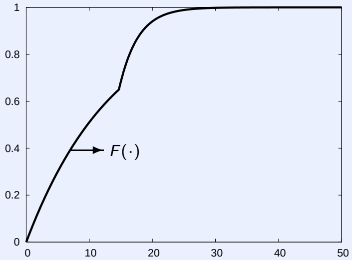
The CEM assumptions:

- If the stress level is fixed, the survivors will fail according to the distribution function of that stress level but starting at previous accumulated fraction failed.

CEM



(a) CDF under different stress level



(b) CDF under CEM

Figure: Example of CEM

Here $F_1(\cdot)$ and $F_2(\cdot)$ are CDF of $Exp(14)$ and $Exp(1)$ respectively.

CEM

Under the assumptions of CEM the CDF of the lifetime is given by

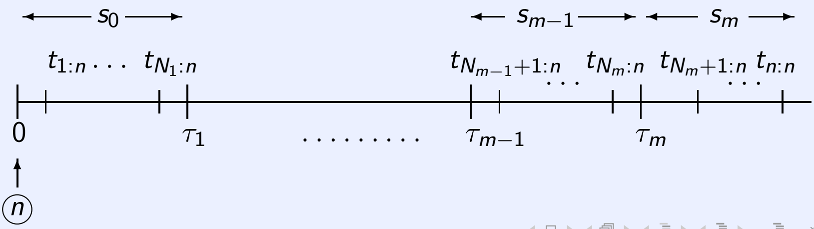
$$F(t) = F_i(t - \tau_{i-1} + h_{i-1}) \quad \text{if } \tau_{i-1} \leq t < \tau_i,\ i = 1,\ 2,\ \ldots,\ m+1,$$

where $\tau_0 = 0$, $\tau_{m+1} = \infty$, $h_0 = 0$ and h_i, $i = 1, 2, \ldots, m$, is the solution of

$$F_{i+1}(h_i) = F_i(\tau_i - \tau_{i-1} + h_{i-1}).$$

Our Model: Form of Data

- Multiple step-stress life test.
- $s_0, s_1, \ldots, s_m$: Stress levels.
- n: Number of items placed in the test.
- τ_i: Time to change stress level form s_{i-1} to s_i.
- n_i: Number of failure at the stress level s_{i-1}.
- $N_i = \sum_{j=1}^{i} n_j$.



Our Model: Distributions

- The lifetime follows an exponential distribution with mean θ_i under stress level s_{i-1}.
- Cumulative exposure model.
- Order restriction: $\theta_1 \geq \ldots \geq \theta_{m+1} > 0$.
- Balakrishnan et. al. (2009)[3]: Using isotonic regression.

[3] Balakrishnan, N., Beutner, E., and Kateri, M. (2009) Order restricted inference for exponential step-stress models, *IEEE Transactions on Reliability*, 58:132-142

Our Model: Reparametrization

- $\theta_2 = \beta_1\theta_1$, where $0 < \beta_1 \leq 1$
- $\theta_3 = \beta_2\theta_2 = \beta_2\beta_1\theta_1$, where $0 < \beta_1,\, \beta_2 \leq 1$.
- In general $\theta_i = \theta_1 \prod_{j=1}^{i-1} \beta_j$, where $0 < \beta_1, \ldots, \beta_m \leq 1$.
- $\{\theta_1, \ldots, \theta_{m+1}\} \xleftrightarrow{\textit{one-to-one}} \{\theta_1, \beta_1, \ldots, \beta_m\}$.

Likelihood Function

- Likelihood function

$$L\left(\theta_1,\, \beta_1,\, \ldots,\, \beta_m;\, \text{Data}\right) \propto \frac{e^{-A(\beta_1,\ldots,\beta_m)/\theta_1}}{\theta_1^{n}\beta_1^{\overline{n}_2}\beta_2^{\overline{n}_3}\ldots\beta_m^{\overline{n}_{m+1}}}.$$

- $A(\beta_1,\, \ldots,\, \beta_m) = D_1 + \sum_{i=2}^{m+1} \frac{D_i}{\prod_{j=1}^{i-1}\beta_j}$.
- For $k = 1,\, 2,\, \ldots,\, m+1$,

$$D_k = \sum_{i=n-\overline{n}_k+1}^{n-\overline{n}_{k+1}} (t_{i:n} - \tau_{k-1}) + \overline{n}_{k+1}(\tau_k - \tau_{k-1})$$

with $\tau_0 = 0$ and $\tau_{m+1} = \infty$.

- $\overline{n}_k = \sum_{i=k}^{m+1} n_i$, $k = 1,\, 2,\, \ldots,\, m+1$. $\overline{n}_{m+2} = 0$.

Maximum Likelihood Estimator

- Maximizing the log-likelihood function over the region $S = (0, \infty) \times (0, 1]^m$.

Maximum Likelihood Estimator

- Maximizing the log-likelihood function over the region $S = (0, \infty) \times (0, 1]^m$.
- The following algorithm can be used to find the MLEs.

Algorithm

Find the estimates $\widehat{\theta}_1^*$, $\widehat{\beta}_1^*$, $\ldots$, $\widehat{\beta}_m^*$ which maximizes log-likelihood function over $(0, \infty)^{m+1}$.

if $\widehat{\beta}_i^* \leq 1$ for all $i = 1, 2, \ldots, m$ **then**

    $\widehat{\theta}_{1,\,\mathrm{MLE}} = \widehat{\theta}_1^*$ and $\widehat{\beta}_{i,\,\mathrm{MLE}} = \widehat{\beta}_i^*$ for all i

else

    while $\widehat{\beta}_i^* > 1$ for at least one i **do**

        replace β_i by 1 in the log-likelihood function for those i for which $\widehat{\beta}_i^* > 1$

        maximize the profile likelihood with respect to the remaining parameters

    end while

    $\widehat{\theta}_{1,\,\mathrm{MLE}} = \widehat{\theta}_1^*$ and $\widehat{\beta}_{i,\,\mathrm{MLE}} = \widehat{\beta}_i^*$ for all i

end if

Remarks

- The algorithm stops in a finite number of steps.

Remarks

- The algorithm stops in a finite number of steps.
- The Algorithm provides the MLEs of θ_1, $\beta_1, \ldots, \beta_m$ under the constraints $0 < \beta_1, \ldots, \beta_m \leq 1$.

Remarks

- The algorithm stops in a finite number of steps.
- The Algorithm provides the MLEs of $\theta_1, \beta_1, \ldots, \beta_m$ under the constraints $0 < \beta_1, \ldots, \beta_m \leq 1$.
- If there is no failure at the stress level s_k for $k = 1, \ldots, m-1$, then the MLE of β_k is 1.

Remarks

- The algorithm stops in a finite number of steps.
- The Algorithm provides the MLEs of θ_1, $\beta_1, \ldots, \beta_m$ under the constraints $0 < \beta_1, \ldots, \beta_m \leq 1$.
- If there is no failure at the stress level s_k for $k = 1, \ldots, m-1$, then the MLE of β_k is 1.
- One can construct asymptotic or bootstrap confidence intervals of the parameters.

Remarks

- The algorithm stops in a finite number of steps.
- The Algorithm provides the MLEs of θ_1, $\beta_1, \ldots, \beta_m$ under the constraints $0 < \beta_1, \ldots, \beta_m \leq 1$.
- If there is no failure at the stress level s_k for $k = 1, \ldots, m-1$, then the MLE of β_k is 1.
- One can construct asymptotic or bootstrap confidence intervals of the parameters.
- Bayesian inference can be performed on the model parameters using a algorithm based on importance sampling.

Question?