Analysis of Simple Step-stress Model in Presence of Competing Risks

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Paper

The talk is based on the following paper.

- Ganguly, A. and Kundu, D. (2016), Analysis of Simple Step-stress Model in Presence of Competing Risks, *Journal of Statistical Computation and Simulation*, 86:1989–2006.

Main Sections

1. Right Censoring Schemes
2. Step-stress Life Test
3. Competing Risks
4. Literature Review
5. Likelihood Inference
6. Interval Estimation
7. Data Analysis
8. Conclusions

Right Censoring Schemes

- Quite useful techniques in reliability life testing.
- Possible termination of experiment before failing all the experimental units.
- Several censoring schemes depending on termination of the experiment.
- Most basic censoring schemes: Type-I and Type-II censoring scheme.

Accelerated Life Tests

- Useful experimental technique to obtain data on the lifetime distribution of highly reliable products.
- Put a sample of products on the test under one or more accelerated stresses to get early failures.
- Need to extrapolate to estimate the lifetime distribution under the normal condition.
- In practice, ALTs are performed in the presence of censoring.

ALT: Examples

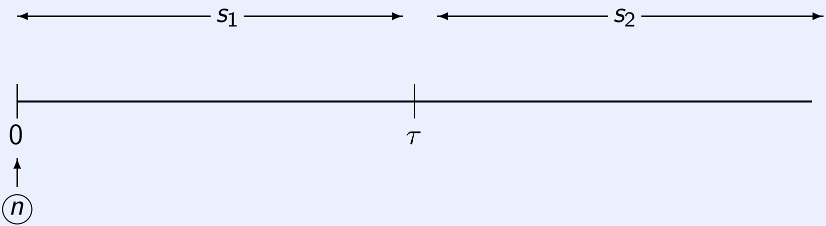
- Food and Drug
  - Performance variables: pH, moisture loss or gain, microbial growth, color, specific chemical reaction.
  - Accelerating variables: Temperature, humidity, chemicals, pH, oxygen, solar radiation.
  - Societies: American Society of Test Methods, US Pharmacopoeia, Pharmaceutical Manufacturers Association.

ALT: Examples

- Nuclear Reactor
  - Performance variables: Strength, creep, creep-rupture.
  - Accelerating variables: Temperature, mechanical stress, contaminants, nuclear radiation.
  - Societies: Institute of Environmental Sciences, American Nuclear Society.

Step-stress Life Tests

- A particular type of accelerated life test.
- Allows to change the stress levels during the life test.
- n : Number of items put on the test.
- s_1, s_2 : Stress levels (Simple SSLT).
- τ : Stress changing time (Pre-fixed).



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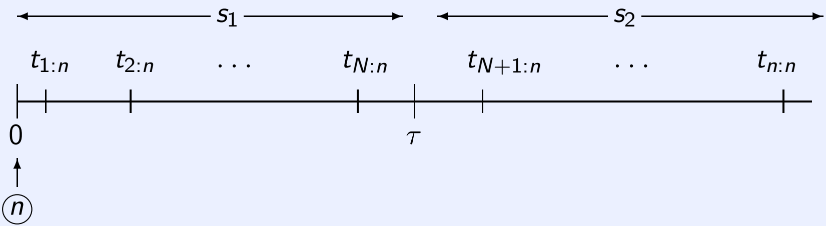
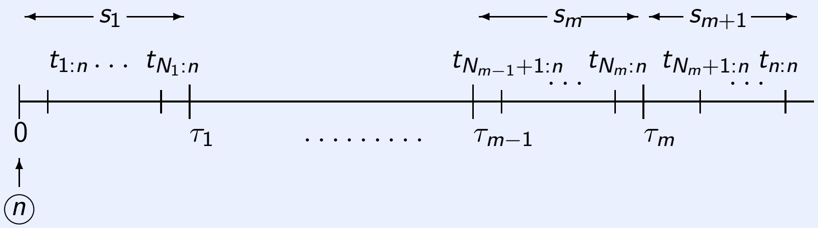


Step-stress Life Tests

- Generalization
  - n : No of items placed on the test.
  - $s_1, s_2, s_3, \ldots, s_{m+1}$: Stress levels.
  - $\tau_1 < \tau_2 < \ldots < \tau_m$: Stress changing times (Pre-fixed).



Advantages

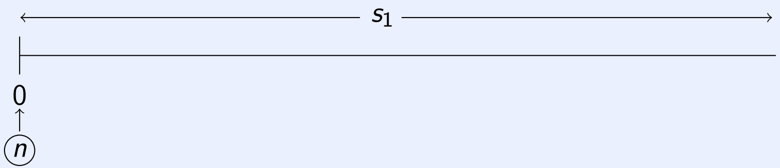
- By increasing the stress level, reasonable number of failure can be obtained.
- Experiment time is reduced.

Disadvantages

- Exact relationship between the stress level and lifetime of the product is needed.
- Model must take into account the effect of stress accumulated.
- Model becomes more complicated.
- Less (no) failures under a stress level.

SSLT with Random Steps

- Change the stress level as soon as pre-specified number of failures occurs.
- n: Number of items put on the test.
- s_1, s_2: Stress levels.
- r: Pre-specified positive integer ($r < n$).



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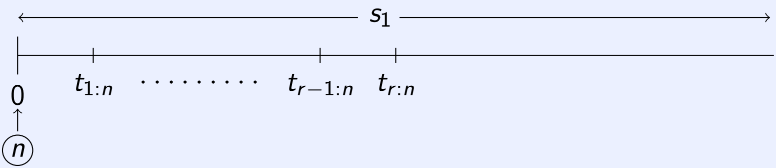

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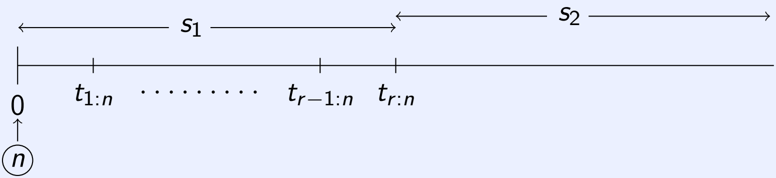

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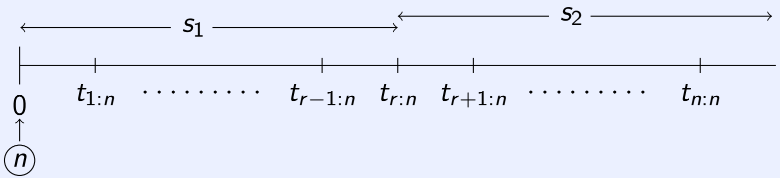

Models

- $F_i(.)$: CDF of lifetime of an item under the stress level s_i, $i = 1, 2, \ldots, m$.
- $F(.)$ is the CDF of lifetime of an item under the step-stress pattern.
- Model needed to relate $F(\cdot)$ to $F_i(\cdot)$, $i = 1, 2, \ldots, m$.
- Popular models
  - Tampered random variable model.
  - Tampered failure rate model.
  - Khamis-Higgins model.
  - Cumulative exposure model.

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  - **Cumulative exposure model**.

CEM for Fixed Step

- Possibly the most popular model.
- First proposed by Seydyakin (1966)[1] and later studied by Nelson (1980)[2] .
- $F_i(\cdot)$ is the CDF of lifetime of an item under the stress level s_i, $i = 1, 2, \ldots, m + 1$.
- $F(\cdot)$ is the CDF of lifetime of an item under the step-stress pattern.

[1] Seydyakin, N. M. (1966) On one physical principle in reliability theory, *Technical Cybernatics*, 3:80-87.

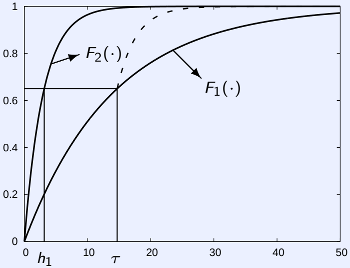
[2] Nelson (1980) Accelerated life testing: step-stress models and data analysis, *IEEE Transactions on Reliability*, 141:288-2838.

CEM for Fixed Step

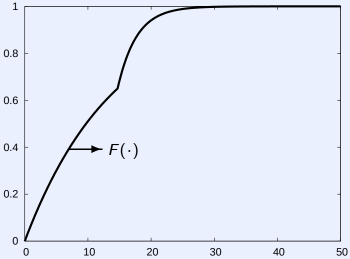
The CEM assumptions:

- If the stress level is fixed, the survivors will fail according to the distribution function of that stress level but starting at previous accumulated fraction failed.

CEM for Fixed Step



(a) CDF under different stress level



(b) CDF under CEM

Figure: Example of CEM

Here $F_1(\cdot)$ and $F_2(\cdot)$ are CDF of $Exp(14)$ and $Exp(1)$ respectively.

CEM for Fixed Step

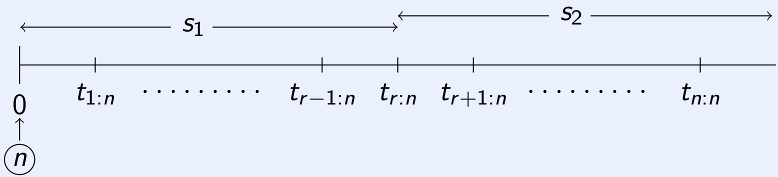
Under the assumptions of CEM the CDF of the lifetime is given by

$$F(t) = F_i(t - \tau_{i-1} + h_{i-1}) \quad \text{if } \tau_{i-1} \leq t < \tau_i,\ i = 1, 2, \ldots, m+1,$$

where $\tau_0 = 0$, $\tau_{m+1} = \infty$, $h_0 = 0$ and h_i, $i = 1, 2, \ldots, m$, is the solution of

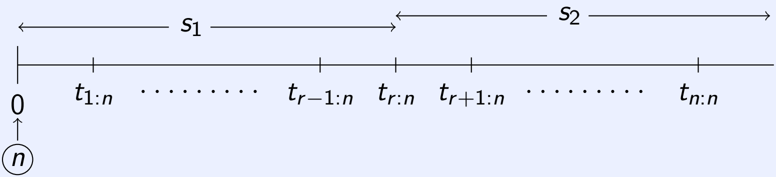
$$F_{i+1}(h_i) = F_i(\tau_i - \tau_{i-1} + h_{i-1}).$$

CEM for SSLT with Random Step



- The distribution of $T_{1:n}, \ldots, T_{r:n}$ is same as that of first r order statistics based on a random sample of size n form the CDF $F_1(\cdot)$.

CEM for SSLT with Random Step



- The distribution of $T_{1:n}, \ldots, T_{r:n}$ is same as that of first r order statistics based on a random sample of size n form the CDF $F_1(\cdot)$.
- The conditional distribution of $T_{r+1:n}, \ldots, T_{n:n}$ given $T_{1:n}, \ldots, T_{r:n}$ is same as that of order statistics based on a random sample of size $n - r$ from the CDF

$$\frac{F_2(t - t_{r:n} + h)}{1 - F_1(t_{r:n})} \quad \text{for } t > t_{r:n},$$

where h is the solution of $F_2(h) = F_1(t_{r:n})$.

CEM for SSLT with Random Step

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  where h is the solution of $F_2(h) = F_1(t_{r:n})$.
- The joint distribution of $T_{1:n}, \ldots, T_{n:n}$ can be found as the multiplication.

Competing Risks

- Suppose that a each transformer fails due to one of the causes, (i) excessive load or (ii) excessive heating.
- The cause of the failure is known along with the failure time.
- This is known as competing risks in statistical literature.

Competing Risks

- There are mainly two ways to analysis the competing risks data.
  - Latent failure times model approach as suggested by Cox (1959)[3].
  - Cause specific hazard function model as suggested by Prentice et al. (1978)[4].

[3] Cox D. R. (1959), The analysis of exponentially distributed lifetimes with two types of failures, *Journal of the Royal Statistical Society Series B*, 21:411–421.

[4] Prentice R. L., Kalbfleish J. D., Peterson Jr. A. V., Flurnoy N., Farewell V. T., Breslow N. E. (1978), The analysis of failure times in presence of competing risks, *Biometrics*, 34:541–554.

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Literature Review

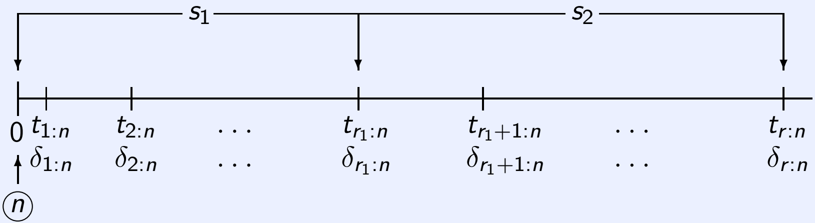
- Xiong and Milliken (1999)[5]: Random stress changing time, exponential CEM, Type-II censoring.
- Balakrishnan and Han (2008)[6]: Pre-fixed stress changing time, exponential CEM, Type-II censoring, competing risks.

[5] Xiong C. and Milliken G. A. (1999) Step-stress life-testing with random stress-change times for exponential data, *IEEE Transaction on Reliability*, 48:141148.

[6] Balakrishnan N. and Han D. (2008) Exact inference for a simple step-stress model with competing risks for failure from exponential distribution under Type-II censoring, *Journal of Statistical Planning and Inference*, 138:41724186

Our Model: Form of Data

- Simple step-stress life test.
- n: Number of items placed in the test.
- r_1, r_2: Two positive integers such that $r = r_1 + r_2 < n$.
- $t_{r_1:n}$: Stress changing time.
- $t_{r:n}$: Experiment termination time.
- Two causes of failures.



Our Model: Distributions

- The latent failure time model assumption of Cox (1959)[7].
- Exponentially distributed latent failure times with following means:

	Cause 1	Cause 2
Stress 1	θ_{11}	θ_{12}
Stress 2	θ_{21}	θ_{22}

- Independently distributed latent failure times.
- CEM assumptions for each of the latent failure distributions.

[7] Cox, D. R. (1959) "The analysis of exponentially distributed lifetimes with two types of failures", *Journal of the Royal Statistical Society*, 21:411–421.

Likelihood Function

- n_i: Number of failures at stress level s_i due to cause one.
- $D_1 = \sum_{i=1}^{r_1} t_{i:n} + (n - r_1)\, t_{r_1:n}$.
- $D_2 = \sum_{i=r_1+1}^{r} (t_{i:n} - t_{r_1:n}) + (n - r)(t_{r:n} - t_{r_1:n})$.
- Likelihood function

$$L(\theta_{11},\, \theta_{12},\, \theta_{21},\, \theta_{22}) \propto \theta_{11}^{-n_1}\, \theta_{12}^{-(r_1-n_1)}\, \theta_{21}^{-n_2}\, \theta_{22}^{-(r_2-n_2)} \times e^{-\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\right)D_1-\left(\frac{1}{\theta_{21}}+\frac{1}{\theta_{22}}\right)D_2}.$$

Maximum Likelihood Estimator

- MLEs of the unknown parameters do not exist if $n_1 = 0$ or $n_1 = r_1$ or $n_2 = 0$ or $n_2 = r_2$.

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$$\widehat{\theta}_{11} = \frac{D_1}{n_1}, \quad \widehat{\theta}_{12} = \frac{D_1}{r_1 - n_1}, \quad \widehat{\theta}_{21} = \frac{D_2}{n_2}, \quad \widehat{\theta}_{22} = \frac{D_2}{r_2 - n_2}.$$

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- Conditional MLEs of the corresponding parameter conditioning on the event $1 \leq n_1 \leq r_1 - 1$ and $1 \leq n_2 \leq r_2 - 1$.

Properties of Estimators

- The conditional distribution of $\widehat{\theta}_{ij}$ can be expressed as a mixture of Gamma distributions.
- The mixing coefficients are the PMF of truncated Binomial distribution.
  - ➤ Conditional moment generating functions of the MLEs are found and then they are inverted to obtain the distributions of MLEs.

Confidence Interval

- $F_{\widehat{\theta}_{ij}}(x, \theta_{ij})$ is a strictly decreasing function of θ_{ij} for all $x > 0$.
- A two-sided $100(1-\alpha)\%$ approximate confidence interval of θ_{ij} is given by $(\theta_{ijL}, \theta_{ijU})$, where θ_{ijL} and θ_{ijU} are the roots of the equations

$$F_{\widehat{\theta}_{ij}}(\widehat{\theta}_{ijobs}, \theta_{ijL}) = 1 - \frac{\alpha}{2} \qquad \text{and} \qquad F_{\widehat{\theta}_{ij}}(\widehat{\theta}_{ijobs}, \theta_{ijU}) = \frac{\alpha}{2},$$

  provided the equations are feasible.
- Parametric bootstrap confidence interval are also considered. It can be obtained in routine manner.

Data Analysis

- Artificially generated: $n = 30$, $r_1 = 10$, $r_2 = 13$, $\theta_{11} = 1$, $\theta_{12} = 1.25$, $\theta_{21} = 0.5$, and $\theta_{22} = 0.675$.
- Number of failures are as follows:

	Cause 1	Cause 2
Stress 1	6	4
Stress 2	5	8

- $\widehat{\theta}_{11} = 1.22$, $\widehat{\theta}_{12} = 1.83$, $\widehat{\theta}_{21} = 0.75$, and $\widehat{\theta}_{22} = 0.47$.

Data Analysis

- Approximate and bootstrap confidence intervals:

	ACI		BCI	
Parameter	LL	UL	LL	UL
θ_{11}	0.666115	2.701967	0.601998	2.488081
θ_{12}	0.876948	5.092153	0.809575	4.998344
θ_{21}	0.388649	1.804309	0.375779	1.863981
θ_{22}	0.279012	0.923804	0.254216	0.861798

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- Performance of bootstrap CI is quite satisfactory.
- Construction of bootstrap CI is quite straight forward.
- Use of bootstrap CI is recommended.

Question?