On Classical and Bayesian Order Restricted Inference for Multiple Exponential Step Stress Model

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Abstract

In this article we consider the multiple step stress model based on the cumulative exposure model assumption. Here, it is assumed that for a given stress level, the lifetime of the experimental units follows exponential distribution and the expected lifetime decreases as the stress level increases. We mainly focus on the order restricted inference of the unknown parameters of the lifetime distributions. First we consider the order restricted maximum likelihood estimators of the model parameters. It is well known that the order restricted maximum likelihood estimators cannot be obtained in explicit forms. We propose an algorithm which stops in finite number of steps and it provides the maximum likelihood estimators. We further consider the Bayes estimates and the associated credible intervals under the squared error loss function. Due to the absence of explicit form of the Bayes estimates, we propose to use the importance sampling technique to compute Bayes estimates. We provide an extensive simulation study in case of three stress levels mainly to see the performances of the proposed methods. Finally the analysis of one real data set has been provided for illustrative purposes.

Key Words: Step-stress Life-tests; cumulative exposure model; Bayes estimates; credible interval; maximum likelihood estimator; confidence interval.

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1 Introduction

Day by day industrial products become highly reliable and as a result it becomes very difficult to get sufficient failure time data during a life testing experiment for any statistical analysis purposes. The accelerated life testing (ALT) procedure is an effective technique used by the experimenters to overcome such difficulties. In an ALT procedure, products are subjected to higher stress level than the normal operating condition, which ensures early failures of the experimental units. Key references on ALT model are Nelson [14] and Bagdanavicius and Nikulin [1], see also Kateri and Kamps [10], Wang et al. [18] and the references cited therein for some recent developments. Various factors such as temperature, voltage, pressure, load etc. are usually used as stress factors, and they can be applied mainly in two different ways. One is known as the constant stress life testing experiment, where the whole sample is divided in to some sub-groups and different stresses are applied to each sub-group separately. Another one is called as the step stress life testing (SSLT) experiment, in which case the stress level gradually increases. In a SSLT, one starts with n number of units at the initial stress level s_0. The stress increases to the next stress level s_1 at a pre-assigned time τ_1 and then at the time point τ_2, the stress level increases to s_2 and so on. In case of only two stress levels the experiment is known as the simple step stress experiment.

Since units are exposed to different stress levels, lifetime distributions of the experimental units vary from one stress level to another. Predominantly used lifetime distributions are exponential, Weibull, generalized exponential, log normal, gamma etc. to analyze data obtained from a SSLT experiment. The most common model assumption made to connect the distributions under different stress levels is the cumulative exposure model (CEM) of Seydyakin [16]. See also Nelson [14] in this respect.

The analysis of SSLT experimental data based on CEM assumption has been discussed quite extensively in the literature. Xiong [19] presented the inference of a simple step stress exponential model by assuming that mean lifetime of the experimental unit is a log-linear function of stresses. Other references on SSLT model are Balakrishnan et al. [4], Balakrishnan and Xie [6, 5], and see the references cited therein. Interested readers are referred to a review article by Balakrishnan [2] on the exact inferential methods of the model parameter of an exponential distribution under different censoring schemes. Recently Mitra et al. [13] have considered the exact inference of a simple step stress model for two parameter exponential distribution. Bayesian inference on SSLT has been considered by Drop et al. [7], Lee and Pan [12], Sha and Pan [17] and Ganguly et al. [8].

To ensure rapid failures of the experimental units, experimenter increases stress levels at the pre-assigned time points in a SSLT experiment. Hence, the expected lifetime of the experimental units gradually decreases with the increase of the stress level. Therefore, there is a natural order restriction among the parameters under different stress levels. If the expected lifetime is θ_i at the stress level s_{i-1}, $i = 1, \ldots, m+1$, then clearly, $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_{m+1}$. Most of the inferences in the literature do not consider this assumption. First Balakrishnan et al. [3] developed order restricted maximum likelihood estimation for the exponential multiple step stress model under Type-I and Type-II censoring scheme using isotonic regression method. Recently Samanta et al. [15] have considered order restricted Bayesian inference for the exponential parameters of a simple step stress model using reparametrization method. A comprehensive review of different aspects of step-stress models can be found in Kundu and Ganguly [11].

Balakrishnan et al. [3] obtained the order restricted maximum likelihood estimators (MLEs) using isotonic regression method and they are quite complicated to use in practice, see for example eqn. (9) of that paper. In this paper we have considered the same model as in Balakrishnan et al. [3], but we have made a reparametrization of the model parameters. First we consider the classical inference of the order restricted parameters. We have provided an algorithm which after finite number of steps produces the MLEs of the unknown parameters in explicit forms. Similarly as in Balakrishnan et al. [3], here also it is observed that the MLEs of the unknown parameters exist even when there is no observation in some of the stress levels, provided they are not on the boundaries. Based on the observed Fisher information matrix, we obtain the confidence intervals of the unknown parameters.

We further consider the order restricted Bayesian inference of the unknown parameters under a fairly flexible priors. Since the Bayes estimates cannot be obtained in explicit forms, we propose to use importance sampling technique to compute the Bayes estimates and the associated credible intervals (CRIs). Extensive simulation experiments have been performed to see the effectiveness of proposed method and its advantages over the unconstrained MLEs. Finally, the analysis of one real data set has been performed for illustrative purposes.

The rest of the paper is organized as follows. In Section 2, we provide the model assumption, the reparametrization and the corresponding likelihood function. In Section 3, the MLEs and the associated theoretical results are provided. Construction of the confidence intervals of the unknown parameters based on the asymptotic distribution of the MLEs and using bootstrap method, have been provided in Section 4. The order restricted Bayesian inference of the unknown parameters is provided in Section 5. The simulation results and the analysis of one real data set have been presented in Section 6, and finally in Section 7 we draw conclusions of the paper.

2 Model Assumption and Likelihood Function

We consider multiple step stress model as in Balakrishnan et al. [3] when the lifetime of the experimental units follow one-parameter exponential distributions. Assume n identical experimental units are put into a life testing experiment with the initial stress level s_0 and the stress changes from s_{i-1} to s_i at the pre-fixed time points $\tau_i, i = 1, \ldots, m$, respectively. The experiment continues till the last unit fails. Hence, the data came from this experiment is of the form $t_{1:n} < \ldots < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < \ldots < t_{n_1+n_2:n} < \tau_2 < \ldots < \tau_m < t_{(n_1+\ldots+n_m+1):n} < \ldots < t_{n:n}$. Here n_i is the number of failures at the stress level s_{i-1} for $i = 1, \ldots, m+1$. Note that some of the n_i may be zero also.

It is further assumed that the lifetime distribution of experimental units under stress level s_{i-1} follows an exponential distribution with mean θ_i. To relate the lifetime distribution of one stress level to the preceding stress level we follow the CEM assumptions, hence, the cumulative distribution function (CDF) of the lifetime of the experimental unit is given by

$$F(t) = \begin{cases} F_1(t) & \text{if } 0 < t \leq \tau_1 \\ F_k(c_{k-1} + t - \tau_{k-1}) & \text{if } \tau_{k-1} < t < \tau_k \\ F_{m+1}(c_m + t - \tau_m) & \text{if } \tau_m < t < \infty, \end{cases} \tag{1}$$

where

$$F_k(x) = 1 - e^{-\frac{x}{\theta_k}}$$

and c_k's can be obtained from the following equations;

$$F_k(c_{k-1}) = F_{k-1}(c_{k-2} + \tau_{k-1} - \tau_{k-2}); \quad k = 2, 3, \ldots, m+1.$$

By solving the above recursion relations one can easily obtain

$$c_{k-1} = \theta_k \sum_{j=1}^{k-1} \frac{\tau_j - \tau_{j-1}}{\theta_j}; \quad k = 2, 3, \ldots, m+1,$$

with $c_0 = 0$ and $\tau_0 = 0$. Hence, the probability density function (PDF) associated with the CDF (1) is given by

$$f(t) = \begin{cases} \frac{1}{\theta_1} e^{-\frac{t}{\theta_1}} & \text{if } 0 < t \leq \tau_1 \\ \frac{1}{\theta_k} e^{-\frac{(c_{k-1}+t-\tau_{k-1})}{\theta_k}} & \text{if } \tau_{k-1} < t < \tau_k \text{ for } k = 2, 3, \ldots, m \\ \frac{1}{\theta_{m+1}} e^{-\frac{(c_m+t-\tau_m)}{\theta_{m+1}}} & \text{if } \tau_m < t < \infty. \end{cases} \tag{2}$$

Since in a step stress experiment as the stress level increases the expected lifetime of the experimental units decreases, there is a natural ordering on θ_i's in this case as follows: $\theta_1 \geq \ldots \geq \theta_{m+1} > 0$. Let us consider the following reparametrization of the parameters:

$$\theta_i = \beta_{i-1}\theta_{i-1} = \theta_1 \prod_{j=1}^{i-1} \beta_j$$

for $i = 2,\ 3,\ \ldots,\ m+1$, where $0 < \beta_1, \ldots, \beta_m \leq 1$. Since there is a one to one correspondence between $\{\theta_1, \ldots, \theta_{m+1}\}$ and $\{\theta_1, \beta_1, \ldots, \beta_m\}$, the statistical inference based on the two sets of parameters will be equivalent.

The likelihood function of the data based on the parameters $\{\theta_1, \beta_1, \ldots, \beta_m\}$, is given by

$$L(\theta_1, \beta_1, \ldots, \beta_m; Data) \propto \frac{e^{-\frac{A(\beta_1,\ldots,\beta_m)}{\theta_1}}}{\theta_1^n \beta_1^{\overline{n}_2} \beta_2^{\overline{n}_3} \ldots \beta_m^{\overline{n}_{m+1}}}. \tag{3}$$

where

$$\begin{aligned}
A(\beta_1, \ldots, \beta_m) &= D_1 + \sum_{i=2}^{m+1} \frac{D_i}{\prod_{j=1}^{i-1} \beta_j}, \\
D_1 &= \sum_{i=1}^{n-\overline{n}_2} t_{i:n} + \overline{n}_2 \tau_1, \\
D_k &= \sum_{i=n-\overline{n}_k+1}^{n-\overline{n}_{k+1}} (t_{i:n} - \tau_{k-1}) + \overline{n}_{k+1}(\tau_k - \tau_{k-1}), \quad k = 2,\, 3,\, \ldots,\, m, \\
D_{m+1} &= \sum_{i=n-\overline{n}_{m+1}+1}^{n} (t_i - \tau_m),
\end{aligned}$$

and $\overline{n}_k = \displaystyle\sum_{i=k}^{m+1} n_i, k = 1, 2, \ldots m+1$. Note that here $A(\beta_1, \ldots, \beta_m)$ depends on the data, but we do not make it explicit for brevity.

3 Maximum Likelihood Estimation

The MLEs of $\{\theta_1, \beta_1, \ldots, \beta_m\}$ can be obtained by maximizing the likelihood function (3) with respect to the unknown parameters over the region $\boldsymbol{S} = (0,\, \infty) \times (0,\, 1]^m$. The log-likelihood function without the additive constant is given by

$$l(\theta_1, \beta_1, \ldots, \beta_m; Data) = -n \ln \theta_1 - \sum_{j=1}^{m} \bar{n}_{j+1} \ln \beta_j - \frac{1}{\theta_1} \left[\sum_{i=1}^{m+1} \frac{D_i}{\prod_{j=0}^{i-1} \beta_j} \right]. \tag{4}$$

Therefore, the MLEs of $\theta_1, \beta_1, \ldots, \beta_m$ can be obtain by maximizing (4) over $\boldsymbol{S}$. Let us first consider the case that there is at least one failure in each stress level, later we will consider the general case. Now, the $m+1$ normal equations can be written as follows:

$$
\begin{aligned}
n\theta_1 &= D_1 + \frac{D_2}{\beta_1} + \frac{D_3}{\beta_1\beta_2} + \ldots + \frac{D_{m+1}}{\beta_1\beta_2\ldots\beta_m}, \\
\overline{n}_2\beta_1 &= \frac{D_2}{\theta_1} + \frac{D_3}{\beta_2\theta_1} + \ldots + \frac{D_{m+1}}{\beta_2\beta_3\ldots\beta_m\theta_1}, \\
&\vdots \\
\overline{n}_m\beta_{m-1} &= \frac{D_m}{\beta_1\beta_2\ldots\beta_{m-2}\theta_1} + \frac{D_{m+1}}{\beta_1\beta_2\ldots\beta_{m-2}\beta_m\theta_1}, \\
\overline{n}_{m+1}\beta_m &= \frac{D_{m+1}}{\beta_1\beta_2\ldots\beta_{m-1}\theta_1}.
\end{aligned}
\tag{5}
$$

Let $\widehat{\theta}_1^*, \widehat{\beta}_1^*, \widehat{\beta}_2^*, \ldots, \widehat{\beta}_m^*$ be the solutions of (5), then

$$
\widehat{\theta}_1^* = \frac{D_1}{n_1}, \quad \widehat{\beta}_1^* = \frac{D_2}{n_2\widehat{\theta}_1^*}, \quad \widehat{\beta}_2^* = \frac{D_3}{n_3\widehat{\beta}_1^*\widehat{\theta}_1^*}, \ldots, \widehat{\beta}_m^* = \frac{D_{m+1}}{n_{m+1}\widehat{\beta}_1^*\widehat{\beta}_2^*\ldots\widehat{\beta}_{m-1}^*\widehat{\theta}_1^*}.
$$

Note that if $0 < \widehat{\beta}_i^* \leq 1$, for $i = 1, 2, \ldots, m$, then $\widehat{\theta}_1^*, \widehat{\beta}_1^*, \ldots, \widehat{\beta}_m^*$, are the MLEs of $\theta_1, \beta_1, \ldots, \beta_m$, respectively. If some of the $\widehat{\beta}_i^* > 1$ for $i = 1, \ldots, m$, the MLEs can be obtained using the following Algorithm 1.

Algorithm 1:

Step 1: For the given data $t = (t_{1:n}, \ldots, t_{n:n})$, estimate $\widehat{\theta}_1^*$, $\widehat{\beta}_1^*, \ldots, \widehat{\beta}_m^*$ and check whether $\widehat{\beta}_i^* \leq 1$ $(i = 1, \ldots, m)$ or not.

Step 2: If $\widehat{\beta}_i^* \leq 1$ for all $i = 1, 2, \ldots, m$, then they are the MLEs of β_i and $\widehat{\theta}_1^*$ is the MLE of θ_1.

Step 3: If one or some of the $\widehat{\beta}_i^* > 1$ then replace all of them by 1 in the log-likelihood function and re-estimate the remaining parameters by maximizing the profile log-likelihood.

Step 4: Check the estimates of β_i's obtain in Step 3 are less than or equals to one or not. If all of them are less than or equals to one then they are the MLEs and the corresponding estimate of θ_1 is the MLE of θ_1.

Step 5: If one or some of the estimates of β_i's obtain in Step 3 is greater than one then repeat Step 3 and Step 4 until we get the estimate of all β_i's less than or equal to 1.

Theorem 1. *Algorithm 1 stops in a finite number of steps.*

Proof. Since m is finite, the algorithm stops in a finite number of steps. $\square$

Theorem 2. *Algorithm 1 provides the MLEs of θ_1, β_1, ..., β_m under the constraints $0 < \beta_1, \ldots, \beta_m \leq 1$.*

Proof. See in the Appendix A. $\square$

Now consider the case when there is no failure in one or more internal stress levels. We will show that in this case also order restricted MLEs exist unlike the general MLEs.

Theorem 3. *If there is no failure at the stress level s_k for $k = 1, \ldots, m-1$, then the MLE of β_k is 1.*

Proof. See in the Appendix B. $\square$

Therefore, using Theorem 3, we have

$$\widehat{\theta}_1^* = \begin{cases} \frac{D_1}{n_1} & \text{if } n_2 \neq 0 \\ \frac{D_1+D_2+\ldots+D_l}{n_1} & \text{if } n_2 = n_3 = \ldots = n_l = 0, \quad n_{l+1} \neq 0, \quad l = 2, 3, \ldots, m, \end{cases}$$

and for $k = 1, 2, \ldots, m-1$.

$$\widehat{\beta}_k^* = \begin{cases} 1 & \text{if } n_{k+1} = 0 \\ \frac{D_{k+1}}{n_{k+1}\widehat{\beta}_1^* \ldots \widehat{\beta}_{k-1}^* \widehat{\theta}_1^*} & \text{if } n_{k+1} \neq 0, \quad n_{k+2} \neq 0 \\ \frac{D_{k+1}+\ldots+D_{k+l}}{n_{k+1}\widehat{\beta}_1^* \ldots \widehat{\beta}_{k-1}^* \widehat{\theta}_1^*} & \text{if } n_{k+1} \neq 0, \quad n_{k+2} = \ldots = n_{k+l} = 0, \quad n_{k+l+1} \neq 0, \quad l = 2, \ldots, m-k, \end{cases}$$
$$\widehat{\beta}_m^* = \frac{D_{m+1}}{n_{m+1}\widehat{\beta}_1^* \widehat{\beta}_2^* \ldots \widehat{\beta}_{m-1}^* \widehat{\theta}_1^*}.$$

Once we obtain $\widehat{\theta_1}^*$, $\widehat{\beta}_1^*, \ldots, \widehat{\beta}_m^*$, the MLEs of θ_1, $\beta_1, \ldots, \beta_m$ can be calculated using Algorithm 1.

4 Different Confidence Intervals

4.1 Asymptotic Confidence Interval

In this section we provide the asymptotic confidence intervals of $\theta_1, \ldots, \theta_{m+1}$ based on the observed Fisher information matrix. First we calculate the observed Fisher information matrix of $\theta_1, \beta_1, \ldots, \beta_m$. Let us denote $\eta_1 = \theta_1$ and $\eta_k = \beta_{k-1}$, for $k = 2, \ldots, m+1$. Then the observed Fisher information matrix becomes

$$F = ((f_{ij})) = ((-\frac{\delta^2 l}{\delta\eta_i \delta\eta_j})).$$

Here for $i = 1, \ldots, m+1$, $j = i+1, \ldots, m+1$

$$\begin{aligned} f_{ii} &= \frac{\overline{n}_i}{\eta_i^2} - \frac{2D_i}{\eta_1\eta_2\ldots\eta_i^3} - \frac{2D_{i+1}}{\eta_1\eta_2\ldots\eta_i^3\eta_{i+1}} - \ldots - \frac{2D_{m+1}}{\eta_1\eta_2\ldots\eta_i^3\eta_{i+1}\ldots\eta_{m+1}}, \\ f_{ij} &= -\frac{D_j}{\eta_1\ldots\eta_{i-1}\eta_i^2\eta_{i+1}\ldots\eta_{j-1}\eta_j^2} - \frac{D_{j+1}}{\eta_1\ldots\eta_{i-1}\eta_i^2\eta_{i+1}\ldots\eta_{j-1}\eta_j^2\eta_{j+1}} \\ &\quad - \ldots - \frac{D_{m+1}}{\eta_1\ldots\eta_{i-1}\eta_i^2\eta_{i+1}\ldots\eta_{j-1}\eta_j^2\eta_{j+1}\ldots\eta_{m+1}} = f_{ji}. \end{aligned}$$

Now to compute the observed Fisher information matrix of $\theta_1, \ldots, \theta_{m+1}$, let us consider the following transformation

$$\widehat{\theta} = \begin{bmatrix} \widehat{\theta}_1 \\ \widehat{\theta}_2 \\ \ldots \\ \widehat{\theta}_m \\ \widehat{\theta}_{m+1} \end{bmatrix} = \begin{bmatrix} \widehat{\theta}_1 \\ \widehat{\beta}_1\widehat{\theta}_1 \\ \ldots \\ \widehat{\beta}_1\widehat{\beta}_2\ldots\widehat{\beta}_{m-1}\widehat{\theta}_1 \\ \widehat{\beta}_1\widehat{\beta}_2\ldots\widehat{\beta}_m\widehat{\theta}_1 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ \ldots \\ g_m \\ g_{m+1} \end{bmatrix}.$$

hence, the observed Fisher information matrix of $\theta_1, \ldots, \theta_{m+1}$ is $G'FG$, where,

$$G' = \begin{bmatrix} \frac{\delta g_1}{\delta\theta_1} & \frac{\delta g_1}{\delta\beta_1} & \ldots & \frac{\delta g_1}{\delta\beta_{m-1}} & \frac{\delta g_1}{\delta\beta_m} \\ \frac{\delta g_2}{\delta\theta_1} & \frac{\delta g_2}{\delta\beta_1} & \ldots & \frac{\delta g_2}{\delta\beta_{m-1}} & \frac{\delta g_2}{\delta\beta_m} \\ \ldots & & & & \\ \frac{\delta g_{m+1}}{\delta\theta_1} & \frac{\delta g_{m+1}}{\delta\beta_1} & \ldots & \frac{\delta g_{m+1}}{\delta\beta_{m-1}} & \frac{\delta g_{m+1}}{\delta\beta_m} \end{bmatrix}$$

and

$$\frac{\delta g_k}{\delta\beta_i} = \begin{cases} \widehat{\beta}_1 \ldots \widehat{\beta}_{i-1}\widehat{\beta}_{i+1} \ldots \widehat{\beta}_{k-1}\widehat{\theta}_1 & \text{for} \quad k = 2, 3, \ldots, m+1 \quad \text{and} \quad i = 1, 2, \ldots, k-1 \\ 0 & \text{for} \quad k = 2, 3, \ldots, m \quad \text{and} \quad i = k, k+1, \ldots, m, \end{cases}$$

$$\frac{\delta g_1}{\delta\theta_1} = 1, \quad \frac{\delta g_1}{\delta\beta_i} = 0; \quad i = 1, \ldots, m, \quad \text{and} \quad \frac{\delta g_k}{\delta\theta_1} = \beta_1\beta_2 \ldots \beta_{k-1}; \quad k = 2, 3, \ldots, m+1.$$

Therefore, under the assumption of asymptotic normality, $100(1-\alpha)\%$ asymptotic CI of θ_i is given by

$$\left[\widehat{\theta}_i \pm z_{1-\frac{\alpha}{2}}\sqrt{V_{ii}}\right],$$

where V_{ii} is (i,i)-th element of the matrix $(G'FG)^{-1}$.

4.2 Bootstrap Confidence Interval

In this subsection, we discuss about the construction of parametric bootstrap confidence intervals of the unknown parameters. The following algorithm can be used to construct the parametric bootstrap confidence intervals.

Algorithm 2:

Step 1: For a given $n, \tau_1, \ldots, \tau_m$ and from the sample $t = (t_{1:n}, t_{2:n}, \ldots, t_{n:n})$ obtain $\widehat{\theta}_1, \ldots, \widehat{\theta}_{m+1}$, the MLEs of the respective parameters.

Step 2: Simulate a sample of size n, say, $t^* = (t^*_{1:n}, t^*_{2:n}, \ldots, t^*_{n:n})$ from (2) with parameters $\widehat{\theta}_1, \widehat{\theta}_2, \ldots \widehat{\theta}_{m+1}$.

Step 3: Using the new sample t^*, estimate MLEs of $\theta_1, \theta_2, \ldots, \theta_{m+1}$, say $\widehat{\theta}_1^{(1)}, \widehat{\theta}_2^{(1)}, \ldots, \widehat{\theta}_{m+1}^{(1)}$.

Step 4: Repeat steps 2 and 3, M times and obtain $\widehat{\theta}_1^{(1)}, \widehat{\theta}_1^{(2)}, \ldots \widehat{\theta}_1^{(M)}, \ldots, \widehat{\theta}_{m+1}^{(1)}, \widehat{\theta}_{m+1}^{(2)}, \ldots, \widehat{\theta}_{(m+1)}^{(M)}$.

Step 5: Arrange $\widehat{\theta}_i^{(1)}, \widehat{\theta}_i^{(2)}, \ldots, \widehat{\theta}_i^{(M)}$, $i = 1, 2, \ldots, m+1$ in ascending order and denote the ordered MLEs as $\widehat{\theta}_i^{[1]}, \widehat{\theta}_i^{[2]}, \ldots, \widehat{\theta}_i^{[M]}$. A two sided $100(1-\alpha)\%$ bootstrap confidence interval of θ_i, $(i = 1, 2, \ldots, m+1)$ is given by $(\widehat{\theta}_i^{[\frac{\alpha}{2}M]}, \widehat{\theta}_i^{[(1-\frac{\alpha}{2})M]})$, here $[x]$ denotes the largest integer less than or equals to x.

5 Bayesian Analysis

It is observed so far that the order restricted MLEs cannot be obtained in explicit forms, hence, the construction of the exact confidence intervals is a difficult task. Therefore, in this case Bayesian inference seems to be a reasonable choice. In this section we consider the Bayesian inference of the unknown parameters and provide the Bayes estimates and the associated CRIs of the model parameters. We use the following notations. An inverted gamma distribution with the parameters $a > 0$ and $b > 0$ will be denoted by $\mathrm{IG}(a, b)$ and it has the PDF

$$f_{IG}(x; a, b) = \frac{a^b}{\Gamma(b)} e^{-\frac{a}{x}} \left(\frac{1}{x}\right)^{b+1}; \quad x > 0,$$

and 0, otherwise. A beta distribution with the parameters $a > 0$ and $b > 0$ will be denoted by $\mathrm{Beta}(a, b)$, and it has the PDF

$$\frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}; \quad 0 < x < 1,$$

and 0, otherwise. The priors of θ_1 and β_i are denoted by $\pi_0(\theta_1)$ and $\pi_i(\beta_i)$, respectively. It is assumed that $\pi_0(\theta_1) \sim \mathrm{IG}(a_0, b_0)$, and $\pi_i(\beta_i) \sim \mathrm{Beta}(a_i, b_i)$ for $i = 1, 2, \ldots, m$, and all the priors are independently distributed. Therefore, the join prior distribution of $\theta_1, \beta_1, \ldots, \beta_m$

is given by

$$\pi(\theta_1, \beta_1, \ldots, \beta_m) \propto e^{-\frac{a_0}{\theta_1}} \left(\frac{1}{\theta_1}\right)^{b_0+1} \beta_1^{a_1-1}(1-\beta_1)^{b_1-1} \ldots \beta_m^{a_m-1}(1-\beta_m)^{b_m-1},$$

for $\theta_1 > 0$ and $0 < \beta_i < 1$, for $i = 1, 2, \ldots, m$. Hence, the posterior distribution of $\theta_1, \beta_1, \beta_2, \ldots, \beta_m$ can be written as

$$\begin{aligned}
\widetilde{\pi}(\theta_1, \beta_1, \ldots, \beta_m | Data) &\propto \beta_1^{a_1-\overline{n}_2-1}(1-\beta_1)^{b_1-1} \times \ldots \times \beta_m^{a_m-\overline{n}_{m+1}-1}(1-\beta_m)^{b_m-1} \times \\
&\quad \left(\frac{1}{\theta_1}\right)^{n+b_0+1} e^{-\frac{1}{\theta_1}(a_0+A(\beta_1, \beta_2, \ldots, \beta_m))} \\
&\propto \widetilde{\pi}_1(\beta_1) \ldots\ldots \widetilde{\pi}_m(\beta_m) \widetilde{\pi}_{m+1}(\theta_1 | \beta_1, \ldots, \beta_m) w_1(\theta_1, \beta_1, \ldots, \beta_m),
\end{aligned}$$

where

$$\begin{aligned}
&\widetilde{\pi}_i(\beta_i) = 1 \text{ for } i = 1, 2, \ldots, m, \\
&\widetilde{\pi}_{m+1}(\theta_1 | \beta_1, \ldots, \beta_m) = \frac{[a_0 + A(\beta_1,\, \beta_2,\, \ldots,\, \beta_m)]^{n+b_0}}{\Gamma(n+b_0)} (\frac{1}{\theta_1})^{n+b_0+1} e^{-\frac{1}{\theta_1}(a_0+A(\beta_1, \beta_2, \ldots, \beta_m))}, \\
&w_1(\theta_1, \beta_1, \ldots, \beta_m) = [a_0 + A(\beta_1,\, \beta_2,\, \ldots,\, \beta_m)]^{-(n+b_0)} \prod_{i=1}^{m} \beta_i^{a_i-\overline{n}_{i+1}-1}(1-\beta_i)^{b_i-1}.
\end{aligned}$$

Hence, under the assumption of the squared error loss function, the Bayes estimate (BE) of any parametric function $g(\theta_1, \beta_1, \ldots, \beta_m)$ is the posterior expectation of $g(\theta_1, \beta_1, \ldots, \beta_m)$ which is given by

$$\begin{aligned}
\widehat{g}_B(\theta_1, \beta_1, \ldots, \beta_m) &= E_{\theta_1, \beta_1, \ldots, \beta_m | Data}\left(g(\theta_1, \beta_1, \ldots, \beta_m)\right) \\
&= \frac{\displaystyle\int_0^1 \cdots \int_0^1 \int_0^\infty g(\theta_1, \beta_1 \ldots, \beta_m) \widetilde{\pi}(\theta_1, \beta_1, \ldots, \beta_m | Data) d\theta_1 d\beta_1 \ldots d\beta_m}{\displaystyle\int_0^1 \cdots \int_0^1 \int_0^\infty \widetilde{\pi}(\theta_1, \beta_1, \ldots, \beta_m | Data) d\theta_1 d\beta_1 \ldots d\beta_m},
\end{aligned} \tag{6}$$

provided the expectation exists. In general, an explicit form of the equation (6) cannot be obtained. Hence, we propose to use the importance sampling method to compute the BE and to construct the associated CRIs. We propose to use the following algorithm to compute the BE and the associated CRIs.

Algorithm 3:

Step 1: Generate $\beta_{11}, \beta_{21}, \ldots, \beta_{m1}$ from Uniform$(0,1)$ distribution.

Step 2: For given $\beta_{11}, \beta_{21}, \ldots, \beta_{m1}$ generate θ_{11} from IG$(n+b_0,\, a_0 + A(\beta_{11}, \ldots, \beta_{m1}))$.

Step 3: Repeat Step 1-Step 2, M times to get $\beta_{11}, \beta_{21}, \ldots, \beta_{m1}, \theta_{11}, \ldots, \beta_{1M}, \beta_{2M}, \ldots, \beta_{mM}, \theta_{1M}$.

Step 4: Compute $g_i = g(\theta_{1i}, \beta_{1i}, \beta_{2i}, \ldots, \beta_{mi}); i = 1, 2, \ldots, M$.

Step 5: Calculate the weights $w_{1i} = \frac{w_1(\theta_{1i}, \beta_{1i}, \beta_{2i}, \ldots, \beta_{mi})}{\sum_{i=1}^{M} w_1(\theta_{1i}, \beta_{1i}, \beta_{2i}, \ldots, \beta_{mi})}$.

Step 6: Compute the BE of $g(\theta_1, \beta_1, \ldots, \beta_m)$ under the squared error loss function as $\widehat{g}(\theta_1, \beta_1, \ldots, \beta_m) = \sum_{j=1}^{M} w_{1j} g_j$.

Step 7: To construct a $100(1-\gamma)\%$ $(0 < \gamma < 1)$ CRI of $g(\theta_1, \beta_1, \ldots, \beta_m)$, first order $g_j's$ for j=1,2,…, M, say $g_{(1)} < g_{(2)} < \ldots < g_{(M)}$ and arrange w_{1j} accordingly to get $w_{1(1)}, w_{1(2)}, \ldots, w_{1(M)}$. Note that $w_{1(1)}, w_{1(2)}, \ldots, w_{1(M)}$ may not be ordered.

Step 8: A $100(1-\gamma)\%$ CRI can be obtain as (g_{j_1}, g_{j_2}) where j_1 and j_2 satisfy

$$j_1,\, j_2 \in \{1,\, 2,\, \ldots,\, M\}\,, \quad j_1 < j_2, \quad \sum_{i=j_1}^{j_2} w_{(i)} \le 1-\gamma < \sum_{i=j_1}^{j_2+1} w_{(i)}. \tag{7}$$

The $100(1-\gamma)\%$ highest posterior density (HPD) CRI of $g(\theta_1, \beta_1, \ldots, \beta_m)$ becomes $\big(g_{(j_1^*)},\, g_{(j_2^*)}\big)$, where $1 \le j_1^* < j_2^* \le M$ satisfy

$$\sum_{i=j_1^*}^{j_2^*} w_{(i)} \le 1-\gamma < \sum_{i=j_1^*}^{j_2^*+1} w_{(i)}, \quad \text{and} \quad g_{(j_2^*)} - g_{(j_1^*)} \le g_{(j_2)} - g_{(j_1)},$$

for all j_1 and j_2 satisfying (7).

6 Simulation and Data Analysis

6.1 Simulation

An extensive simulation study has been performed for multiple (3 steps) step stress model to see the effectiveness of the proposed methods. In the simulation study we have taken $\theta_1 = 10$, $\theta_2 = 5$, and $\theta_3 = 3$, different sample sizes (n) and different τ_1 and τ_2 values.

We have computed the order restricted MLEs of the unknown parameters and the associated mean squared errors (MSEs). We have obtained the asymptotic and bootstrap confidence intervals of these parameters based on order restricted MLEs. For comparison purposes, we have computed the MLEs with out any order restriction, the associated MSEs, and also the asymptotic and bootstrap confidence intervals based on them. Further, in each case the Bayes estimates and the associated CRIs are obtained. We have used non-informative priors and we have taken the hyper parameter values as follows: $a_0 = 0.0001$, $b_0 = 0.0001$, $a_1 = 1$, $b_1 = 1$, $a_2 = 1$ and $b_2 = 1$. In each case we have computed the Bayes estimates and the associated highest posterior density (HPD) and symmetric CRIs based on the importance sampling technique.

We have obtained the average estimates (AEs) and the associated MSEs of all the three different estimates. The results are reported in Table 1. In case of the confidence and credible intervals we have obtained the average lengths (ALs) and the associated coverage percentages (CPs) for 95% level of confidence and the results are reported in Table 2. In all the above cases results are based on 1000 replications. At each replication, the Bayes estimates and both the credible intervals are obtained based on 5000 importance samples.

Some of the points are quite clear from the experimental results. The performances of both the MLEs and the Bayes estimators with respect to the non-informative priors are quite satisfactory. It is observed that for all the parameters in all the three cases as the sample size increases the biases and the MSEs decrease. It indicates the consistency behavior of the estimators. The average biases and the MSEs of the order restricted MLEs are smaller

Table 1: Average values and the corresponding mean squared errors of the different estimators when $\theta_1 = 10$, $\theta_2 = 5$, $\theta_3 = 3$.

Order restricted MLEs

			θ_1		θ_2		θ_3	
n	τ_1	τ_2	AE	MSE	AE	MSE	AE	MSE
20	4	7	11.65	40.54	5.50	4.86	2.85	1.03
20	4	9	11.52	34.35	5.32	3.27	2.81	1.43
20	6	8	11.00	21.29	5.76	8.01	2.94	1.09
30	4	7	11.09	17.33	5.43	3.86	2.94	0.71
30	4	9	11.04	16.52	5.24	2.18	2.87	1.01
30	6	8	10.74	9.88	5.51	5.05	2.91	0.70
40	4	7	10.82	11.87	5.31	2.41	2.96	0.54
40	4	9	10.75	12.26	5.24	1.91	2.99	0.84
40	6	8	10.30	6.56	5.57	4.42	2.98	0.60
50	4	7	10.44	7.02	5.26	2.01	2.99	0.46
50	4	9	10.51	9.55	5.17	1.27	2.98	0.64
50	6	8	10.44	5.49	5.39	3.37	2.96	0.46

MLEs without order restriction

			θ_1		θ_2		θ_3	
n	τ_1	τ_2	AE	MSE	AE	MSE	AE	MSE
20	4	7	11.45	43.27	5.90	15.58	3.07	1.50
20	4	9	11.74	41.26	5.45	5.46	2.99	2.04
20	6	8	11.05	28.40	6.36	20.72	3.07	1.44
30	4	7	11.23	26.00	5.53	5.52	3.02	0.87
30	4	9	10.94	17.15	5.26	3.08	2.98	1.32
30	6	8	10.67	10.61	6.02	15.98	3.00	0.90
40	4	7	10.76	13.73	5.43	3.61	3.04	0.61
40	4	9	10.77	13.74	5.16	1.66	3.00	0.98
40	6	8	10.37	7.69	5.56	6.21	3.00	0.64
50	4	7	10.51	8.35	5.22	2.15	3.02	0.50
50	4	9	10.55	10.25	5.13	1.42	3.02	0.79
50	6	8	10.48	5.65	5.71	9.23	2.98	0.50

Order restricted Bayes estimators

			θ_1		θ_2		θ_3	
n	τ_1	τ_2	AE	MSE	AE	MSE	AE	MSE
20	4	7	12.63	47.67	5.83	3.51	3.18	0.80
20	4	9	12.63	46.35	5.62	2.98	3.05	0.97
20	6	8	11.83	20.77	6.06	4.49	3.17	0.94
30	4	7	11.54	29.24	5.62	2.40	3.13	0.63
30	4	9	11.44	17.45	5.57	2.03	3.14	0.72
30	6	8	11.00	10.58	5.79	2.83	3.16	0.62
40	4	7	10.96	10.75	5.56	1.87	3.14	0.49
40	4	9	10.97	11.69	5.43	1.42	3.13	0.57
40	6	8	10.94	8.52	5.86	2.77	3.16	0.46
50	4	7	10.92	9.22	5.52	1.67	3.10	0.40
50	4	9	10.95	9.81	5.34	1.06	3.11	0.50
50	6	8	10.69	6.22	5.69	2.12	3.15	0.43

than the corresponding Bayes estimates in case of θ_1, but they are larger for θ_2 and θ_3 for all sample sizes and for all τ_1, τ_2.

Comparing the two confidence intervals in case of order restricted MLEs, it is observed

Table 2: Average lengths and coverage percentages of different 95% confidence and credible intervals of θ_1, θ_2 and θ_3.

Confidence intervals based on order restricted MLEs.

			Bootstrap CI						Asymptotic CI					
			θ_1		θ_2		θ_3		θ_1		θ_2		θ_3	
n	τ_1	τ_2	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
20	4	7	30.4	92.6	9.0	96.7	3.7	92.0	19.4	93.3	9.4	93.9	4.2	88.3
20	4	9	29.9	91.7	7.3	96.3	4.1	88.4	20.3	93.8	7.4	93.9	5.0	82.9
20	6	8	22.7	92.7	10.2	98.7	3.8	89.7	15.5	92.4	12.7	93.0	4.3	89.8
30	4	7	20.4	95.1	7.0	97.0	3.2	91.5	15.0	94.8	7.4	94.6	3.5	90.4
30	4	9	20.3	95.8	5.8	96.4	3.6	89.7	15.4	92.4	6.0	93.7	4.2	87.4
30	6	8	14.9	94.6	8.7	98.2	3.2	92.1	11.9	94.5	10.1	93.3	3.5	90.0
40	4	7	15.4	95.4	6.1	96.6	2.8	93.9	12.5	94.6	6.3	94.6	3.0	90.8
40	4	9	15.7	93.9	5.0	95.3	3.3	92.1	12.4	94.3	5.2	94.9	3.7	89.9
40	6	8	11.3	95.5	7.4	96.7	2.8	92.5	9.9	94.6	8.9	94.6	3.0	92.7
50	4	7	12.8	94.8	5.3	96.1	2.6	93.4	10.6	95.3	5.3	92.8	2.8	92.3
50	4	9	12.8	93.5	4.5	94.7	3.0	92.6	10.5	95.1	4.5	95.7	3.4	91.9
50	6	8	9.7	95.0	6.7	98.1	2.6	93.7	8.8	94.9	7.4	94.5	2.7	92.7

Confidence intervals based on MLEs without any order restriction.

			Bootstrap CI						Asymptotic CI					
			θ_1		θ_2		θ_3		θ_1		θ_2		θ_3	
n	τ_1	τ_2	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
20	4	7	30.2	92.6	15.4	95.1	4.7	90.8	20.7	90.7	10.6	91.5	4.4	88.1
20	4	9	30.6	93.0	11.0	95.3	6.1	91.3	21.4	92.2	7.9	93.2	5.8	87.2
20	6	8	22.9	93.1	16.1	95.1	4.8	93.3	15.5	93.1	16.6	89.3	4.6	88.6
30	4	7	22.9	93.3	10.8	94.5	3.7	93.8	14.9	92.5	8.0	92.4	3.6	90.6
30	4	9	20.8	94.2	7.3	94.6	4.6	89.7	15.7	93.5	6.1	91.7	4.6	89.2
30	6	8	14.6	94.1	16.5	94.0	3.8	93.0	11.9	94.8	12.9	91.6	3.6	89.6
40	4	7	15.6	94.2	8.3	95.1	3.1	94.5	12.1	92.5	6.4	94.0	3.1	93.0
40	4	9	16.1	93.7	5.8	95.1	3.9	92.2	12.3	93.3	5.2	93.7	3.9	89.5
40	6	8	11.3	95.4	14.8	93.4	3.2	92.8	9.9	94.8	9.3	92.2	3.1	91.0
50	4	7	13.0	94.0	6.7	93.7	2.8	93.5	10.5	94.7	5.6	95.0	2.8	92.3
50	4	9	12.9	94.5	4.9	94.9	3.6	95.5	10.5	92.8	4.5	94.0	3.5	91.7
50	6	8	9.7	95.9	12.0	94.3	2.8	93.0	8.8	94.1	8.0	91.6	2.8	91.8

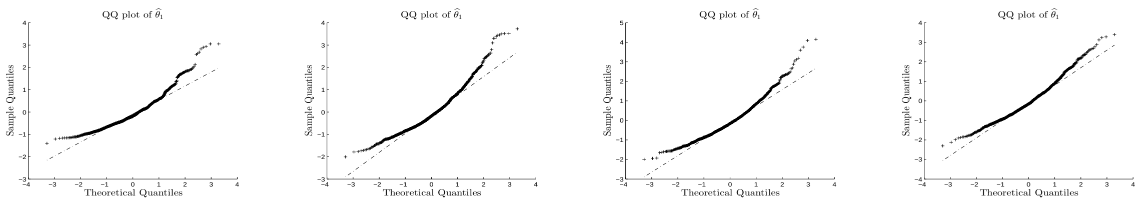
Credible intervals.

			Symmetric CRI						HPD CRI					
			θ_1		θ_2		θ_3		θ_1		θ_2		θ_3	
n	τ_1	τ_2	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
20	4	7	22.2	97.1	7.6	97.4	4.0	97.6	19.0	97.5	7.1	97.6	3.8	96.0
20	4	9	22.4	97.2	6.7	94.7	4.3	96.5	19.0	97.4	6.2	96.1	4.0	94.5
20	6	8	16.8	96.9	8.8	98.5	4.1	96.2	15.1	97.2	8.1	98.7	3.9	94.9
30	4	7	15.7	96.1	6.2	97.2	3.3	96.1	14.1	95.8	5.8	97.2	3.2	95.9
30	4	9	15.3	96.1	5.5	96.6	3.7	96.8	13.8	96.0	5.2	97.7	3.6	96.2
30	6	8	12.2	96.1	7.1	98.3	3.4	96.2	11.3	95.8	6.7	98.4	3.3	95.7
40	4	7	12.4	95.8	5.4	97.1	2.9	96.1	11.5	95.7	5.2	97.1	2.8	94.8
40	4	9	12.4	95.8	4.7	96.0	3.3	97.1	11.5	96.5	4.5	96.4	3.2	96.6
40	6	8	10.5	95.1	6.6	97.7	3.0	97.2	9.9	96.5	6.3	98.5	2.9	97.2
50	4	7	11.0	95.5	5.0	97.1	2.6	96.0	10.3	96.5	4.8	96.8	2.5	95.1
50	4	9	11.1	95.2	4.2	97.2	3.0	96.5	10.4	95.5	4.1	97.3	2.9	95.3
50	6	8	9.1	94.8	5.9	97.9	2.7	95.8	8.6	95.8	5.7	97.8	2.6	95.1

that both of them perform quite well. For both the confidence intervals the average lengths decrease as the sample size increases. Between the two confidence intervals, the bootstrap CIs are preferable than the asymptotic CIs in terms of the coverage percentages although in certain cases the average lengths are slightly longer. Now comparing the two credible intervals it is observed that the HPD CRIs perform slightly better than the symmetric CRIs. In all the cases the coverage percentages are very close to the corresponding nominal values. Finally comparing the bootstrap CIs and the HPD CRIs it is observed that HPD CRIs perform slightly better than the bootstrap CIs in terms of lower average lengths.

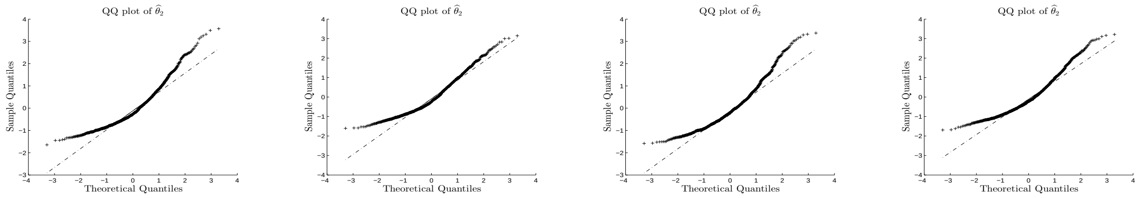
Now comparing the order restricted MLEs with the unrestricted MLEs, it has been observed that the performance of order restricted MLEs are better than the unrestricted MLEs, in terms of MSE, specially for θ_2 and θ_3. Since in both the cases the estimates of θ_1 are same, their performances are also very similar. The average lengths of CIs for θ_2 and θ_3 are lower when they are computed based on order restricted MLEs.

It may be mentioned that to construct the asymptotic confidence intervals it has been assumed that the MLEs are asymptotically normally distributed. To check the validity of that assumption, we have provided the quantile-quantile (QQ) plots of $\widehat{\theta}_1$, $\widehat{\theta}_2$ and $\widehat{\theta}_3$ for different values of n in Figure 1 to Figure 3. It has been observed that the theoretical quantile and the observed quantile converge as n increases. The performances of the asymptotic confidence intervals are quite good even for moderate sample sizes.



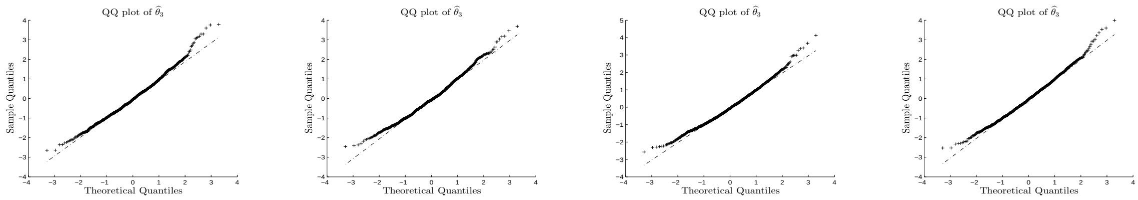
(a) $n = 20$ (b) $n = 30$ (c) $n = 40$ (d) $n = 50$

Figure 1: QQ Plots of $\widehat{\theta}_1$ with parameter values $\theta_1 = 10$, $\theta_2 = 5$, $\theta_3 = 3$, and for $\tau_1 = 6$, $\tau_2 = 8$.



(a) $n = 20$ (b) $n = 30$ (c) $n = 40$ (d) $n = 50$

Figure 2: QQ Plots of $\widehat{\theta}_2$ with parameter values $\theta_1 = 10$, $\theta_2 = 5$, $\theta_3 = 3$, and for $\tau_1 = 6$, $\tau_2 = 8$.



(a) $n = 20$ (b) $n = 30$ (c) $n = 40$ (d) $n = 50$

Figure 3: QQ Plots of $\widehat{\theta}_3$ with parameter values $\theta_1 = 10$, $\theta_2 = 5$, $\theta_3 = 3$, and for $\tau_1 = 6$, $\tau_2 = 8$.

6.2 Fish Data Set

Here we analyze a multiple step stress data set taken from Greven et al. [9]. Here a group of 15 fishes have been taken to observe their swimming performances. Fish were swum at initial flow rate 15 cm/sec., and then flow rate was increased by 5 cm/sec. at time 110, 130, 150, 170 minutes. Here flow rate has been considered as stress factor. The time at which a fish could not maintain its position is called fatigue time and is recorded as the failure data. The flow rate was increased four times, therefore, we have five stress levels. The observed failure data are: 91.00, 93.00, 94.00, 98.20, 115.81, 116.00, 116.50, 117.25, 126.75, 127.50, 154.33, 159.50, 164.00, 184.14, 188.33. Number of failure at the first, second, third, fourth and fifth stress level are 4, 6, 0, 3, 2, respectively.

We analyze this data set assuming multiple exponential multiple step stress model with the order restriction. First we have considered the Bayesian analysis of the data set by considering non-informative prior assumption. Bayes estimates of θ_1, θ_2, θ_3, θ_4, θ_5 under square error loss function are 414.95, 69.082, 52.082, 29.337, 17.89, respectively. We provide symmetric and HPD CRIs in Table 3.

Table 3: Symmetric and highest posterior density credible intervals of the different parameters of the fish data set.

		θ_1		θ_2		θ_3		θ_4		θ_5	
CI	Level	LL	UL	LL	UL	LL	UL	LL	UL	LL	UL
Symmetric	90%	165.44	837.71	31.32	137.50	25.77	103.45	14.52	54.53	7.16	32.85
	95%	153.41	982.08	28.76	157.66	22.71	121.19	13.43	60.36	6.16	38.39
	99%	124.03	1449.60	25.48	220.03	19.60	161.99	10.18	87.26	4.91	51.18
HPD	90%	143.13	702.53	26.24	115.87	21.14	87.90	10.08	44.94	5.25	28.45
	95%	128.56	842.69	25.48	141.01	19.60	106.00	11.53	56.67	4.54	32.97
	99%	94.59	1210.60	21.69	188.76	16.38	141.50	7.49	72.54	3.65	45.72

We have also obtained the order restricted MLEs and CIs of model parameters for the same data set. Note that since there is no failure at the 3-rd stress level, without the ordered restricted assumptions the MLEs of the unknown parameters do not exist. The order restricted MLEs of θ_1, θ_2, θ_3, θ_4, θ_5 are 396.55, 43.302, 43.302, 22.61, 16.235, respectively. Asymptotic and bootstrap CIs of parameters are given in Table 4.

Table 4: Asymptotic and Bootstrap confidence intervals of the different parameters of the fish data set.

		θ_1		θ_2		θ_3		θ_4		θ_5	
CI	Level	LL	UL	LL	UL	LL	UL	LL	UL	LL	UL
Asymptotic	90%	71.38	721.72	11.04	75.56	6.53	80.07	1.20	44.02	0.00	35.06
	95%	7.93	785.17	4.75	81.86	0.00	87.25	0.00	48.19	0.00	38.74
	99%	0.00	906.12	0.00	93.86	0.00	100.92	0.00	56.16	0.00	45.74
Bootstrap	90%	207.96	1582.20	23.99	133.95	19.97	70.86	23.99	133.95	19.97	70.86
	95%	182.63	1612.80	21.33	202.03	17.90	82.72	21.33	202.03	17.90	82.72
	99%	148.21	1642.70	17.34	239.26	15.07	112.31	17.34	239.26	15.07	112.31

Now the natural question is about the goodness of fit of the proposed model to the above data set. We calculate the Kolmogorov-Smirnov (KS) distance between the empirical distribution function (EDF) and the fitted distribution function (FDF) and also obtain the associate p-value. The KS distance and the associated p value between the EDF and the FDF based on the Bayes estimates are 0.2622 and 0.2539, respectively. Similarly, the KS distance and the associated p value between the EDF and the FDF based on the MLEs are 0.2051 and 0.5536, respectively. Therefore, based on the KS distances and the associated p values, it can be said that the order restricted multiple exponential step stress model fits the data quite well and in this case MLEs are preferable compared to the Bayes estimates based on the non-informative priors.

7 Conclusion

In this paper we have considered the order restricted inference of the multiple exponential step stress model. This problem was first considered by Balakrishnan et al. [3] and they obtained the MLEs of the unknown parameters based on isotonic regression method. The main contribution of this paper is that we have provided the solution in a simpler manner. We have considered the Bayesian inference and we have suggested to compute the Bayes estimates and the associated credible intervals of the unknown parameters based on importance sampling technique. The MLEs of the unknown parameters can be obtained in explicit forms. We have performed some simulation experiments and it is observed that the Bayes estimates and MLEs work quite well. One real data set has been analyzed and it is observed that the proposed model fits the real data set quite well. It may be mentioned that our method can be extended for other lifetime distributions also. Work is in progress and it will be reported later.

Acknowledgements:

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A Appendix

Proof of Theorem 2:

We provide the proof for four stress levels because of brevity. The same argument holds for the general case. For $m = 3$, the log-likelihood function becomes

$$l(\theta_1, \beta_1, \beta_2, \beta_3) = -n \ln \theta_1 - \bar{n}_2 \ln \beta_1 - \bar{n}_3 \ln \beta_2 - \bar{n}_4 \ln \beta_3 - \frac{1}{\theta_1} \left[D_1 + \frac{D_2}{\beta_1} + \frac{D_3}{\beta_1 \beta_2} + \frac{D_4}{\beta_1 \beta_2 \beta_3} \right]. \tag{8}$$

The function (8) has a unique maximum at $(\theta_1^*, \beta_1^*, \beta_2^*, \beta_3^*)$, where

$$\theta_1^* = \frac{D_1}{n_1}, \quad \beta_1^* = \frac{D_2 n_1}{D_1 n_2} \quad \beta_2^* = \frac{D_3 n_2}{D_2 n_3}, \quad \beta_3^* = \frac{D_4 n_3}{D_3 n_4}. \tag{9}$$

Moreover, the function (8) does not have any other local maximum. Observe that for a given $(\beta_1, \beta_2, \beta_3)$, the function (8) attains its maximum when

$$\widehat{\theta}_1(\beta_1, \beta_2, \beta_3) = \frac{1}{n}\left[D_1 + \frac{D_2}{\beta_1} + \frac{D_3}{\beta_1\beta_2} + \frac{D_4}{\beta_1\beta_2\beta_3}\right].$$

Substituting $\widehat{\theta}(\beta_1, \beta_2, \beta_3)$ in (8) and ignoring the additive constant the profile log-likelihood function of $\beta_1, \beta_2, \beta_3$ can be obtained as

$$p(\beta_1, \beta_2, \beta_2) = -n \ln\left[D_1 + \frac{D_2}{\beta_1} + \frac{D_3}{\beta_1\beta_2} + \frac{D_4}{\beta_1\beta_2\beta_3}\right] - \bar{n}_2 \ln \beta_1 - \bar{n}_3 \ln \beta_2 - \bar{n}_4 \ln \beta_3. \tag{10}$$

Hence,

$$\sup_{\substack{\theta_1 \geq 0 \\ 0 \leq \beta_1, \beta_2, \beta_3 \leq 1}} l(\theta_1, \beta_1, \beta_2, \beta_3) = \sup_{0 \leq \beta_1, \beta_2, \beta_3 \leq 1} p(\beta_1, \beta_2, \beta_3).$$

From (10) it is observed that the function $p(\beta_1, \beta_2, \beta_3)$ has a unique maximum at $(\beta_1^*, \beta_2^*, \beta_3^*)$, where β_1^*, β_2^* and β_3^*, are same as defined in (9), and the function (10) does not have any other local maximum.

Now we claim that if $\beta_1^* > 1$, then

$$\sup_{\substack{0 \leq \beta_1 \leq 1 \\ \beta_2 \geq 0, \beta_3 \geq 0}} p(\beta_1, \beta_2, \beta_3) = \sup_{\beta_2 \geq 0, \beta_3 \geq 0} p(1, \beta_2, \beta_3). \tag{11}$$

Suppose (11) is not true, then there exists $0 < \widetilde{\beta}_1 < 1$, $\widetilde{\beta}_2 > 0$ and $\widetilde{\beta}_3 > 0$, such that

$$\sup_{\substack{0 \leq \beta_1 \leq 1 \\ \beta_2 \geq 0, \beta_3 \geq 0}} p(\beta_1, \beta_2, \beta_3) = p(\widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_3).$$

It implies $(\widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_3) \neq (\beta_1^*, \beta_2^*, \beta_3^*)$ is a local maximum of $p(\beta_1, \beta_2, \beta_3)$ as $p(\beta_1,\ \beta_2,\ \beta_3) \to -\infty$ as $\beta_2 \to \infty$ and $p(\beta_1,\ \beta_2,\ \beta_3) \to -\infty$ as $\beta_3 \to \infty$. Clearly it is a contradiction.

Along the same line it follows that if $\beta_2^* > 1$, then

$$\sup_{\substack{0\le\beta_2\le 1\\ \beta_1\ge 0,\beta_3\ge 0}} p(\beta_1,\beta_2,\beta_3) = \sup_{\beta_1\ge 0,\beta_3\ge 0} p(\beta_1,1,\beta_3). \tag{12}$$

and if $\beta_3^* > 1$, then

$$\sup_{\substack{0\le\beta_3\le 1\\ \beta_1\ge 0,\beta_2\ge 0}} p(\beta_1,\beta_2,\beta_3) = \sup_{\beta_1\ge 0,\beta_2\ge 0} p(\beta_1,\beta_2,1).$$

Combining (11) and (12) we can obtain if $\beta_1^* > 1$ and $\beta_2^* > 1$ then

$$\sup_{\substack{0\le\beta_1,\beta_2\le 1\\ \beta_3\ge 0}} p(\beta_1,\beta_2,\beta_3) = \sup_{\beta_3\ge 0} p(1,1,\beta_3).$$

Similarly, if $\beta_1^* > 1$ and $\beta_3^* > 1$, then

$$\sup_{\substack{0\le\beta_1,\beta_3\le 1\\ \beta_2\ge 0}} p(\beta_1,\beta_2,\beta_3) = \sup_{\beta_2\ge 0} p(1,\beta_2,1),$$

if $\beta_2^* > 1$ and $\beta_3^* > 1$, then

$$\sup_{\substack{0\le\beta_2,\beta_3\le 1\\ \beta_1\ge 0}} p(\beta_1,\beta_2,\beta_3) = \sup_{\beta_1\ge 0} p(\beta_1,1,1)$$

and if $\beta_1^* > 1$, $\beta_2^* > 1$ and $\beta_3^* > 1$, then

$$\sup_{0\le\beta_1,\beta_2,\beta_3\le 1} p(\beta_1,\beta_2,\beta_3) = p(1,1,1).$$

Further observe that

$$\sup_{0\le\beta_1,\beta_2,\beta_3\le 1} p(\beta_1,\beta_2,\beta_3) = \sup_{0\le\beta_i\le 1}\ \sup_{0\le\beta_j\le 1}\ \sup_{0\le\beta_k\le 1} p(\beta_1,\beta_2,\beta_3), \tag{13}$$

for all $i \ne j \ne k$ and $1 \le i,j,k \le 3$.

Now we consider different cases.

Case 1: $\widehat{\beta}_1^* > 1, \widehat{\beta}_2^* > 1, \widehat{\beta}_3^* > 1$. The MLEs of β_1, β_2 and β_3 are 1, 1 and 1, respectively.

Case 2: $\widehat{\beta}_1^* > 1, \widehat{\beta}_2^* > 1, \widehat{\beta}_3^* \leq 1$. The MLEs of β_1 and β_2 are 1 and 1, respectively, and the MLEs of β_3 can be obtained as the $\arg\max_{0 \leq \beta_3 \leq 1} p(1, 1, \beta_3)$. Since $p(1, 1, \beta_3)$ is an unimodal function, it has a unique maximum. Further, $\sup\limits_{0 \leq \beta_1, \beta_2, \beta_3 \leq 1} p(\beta_1, \beta_2, \beta_3) = \sup\limits_{0 \leq \beta_3 \leq 1} p(1, 1, \beta_3)$ due to (13).

Case 3: $\widehat{\beta}_1^* > 1, \widehat{\beta}_2^* \leq 1, \widehat{\beta}_3^* \leq 1$. The MLEs of β_1 is 1 and the MLEs of β_2 and β_3 can be obtained as the $\arg\max_{0 \leq \beta_2, \beta_3 \leq 1} p(1, \beta_2, \beta_3)$. The function $p(1, \beta_2, \beta_3)$ has a unique maximum and we repeat the same argument as before.

The other cases can be considered along the same line. This proves Theorem 2. □

B Appendix

Proof of Theorem 3:

We will provide the proof mainly for five stress level, although the same proof holds for the general case also. We will not consider the cases where any one or both of n_1 and n_5 is zero, since if $n_1 = 0$ or $n_5 = 0$, the MLEs of all the parameters do not exists.

Case 1: Exactly one internal stress level with zero failure.
Here we consider $m = 4$ and without loss of generality let $n_3 = 0$. The log-likelihood function is given by

$$\begin{aligned} l(\theta_1, \beta_1, \beta_2, \beta_3, \beta_4) &= -n \ln \theta_1 - (n_2 + n_4 + n_5) \ln \beta_1 - (n_4 + n_5) \ln \beta_2 - (n_4 + n_5) \ln \beta_3 \\ &\quad -n_5 \ln \beta_4 - \frac{D_1}{\theta_1} - \frac{D_2}{\beta_1 \theta_1} - \frac{D_3}{\beta_1 \beta_2 \theta_1} - \frac{D_4}{\beta_1 \beta_2 \beta_3 \theta_1} - \frac{D_5}{\beta_1 \beta_2 \beta_3 \beta_4 \theta_1}. \qquad (14) \end{aligned}$$

It can be easily shown as in Theorem 1, that the function (14) has a unique global maximum and it does not have any local maximum. For a fixed β_2, the function (14) is maximized when

$$\widehat{\theta}_1 = \frac{D_1}{n_1}, \quad \widehat{\beta}_1(\beta_2) = \frac{n_1(\beta_2 D_2 + D_3)}{n_2 \beta_2 D_1}, \quad \widehat{\beta}_3(\beta_2) = \frac{n_2 D_4}{n_4(\beta_2 D_2 + D_3)}, \quad \widehat{\beta}_4(\beta_2) = \frac{n_4 D_5}{n_5 D_4}.$$

Therefore, the profile log-likelihood of β_2 without the additive constant is given by

$$l_2(\beta_2) = n_2 \ln \beta_2 - n_2 \ln(\beta_2 D_2 + D_3).$$

Since,

$$\frac{dl_2(\beta_2)}{d\beta_2} = \frac{n_2 D_3}{\beta_2 D_2 + D_3} \geq 0,$$

the profile log-likelihood of β_2 is an increasing function of β_2 ($0 < \beta_2 \leq 1$). Hence, the maximum occurs at $\beta_2 = 1$. Therefore,

$$\sup_{\substack{\theta_1 \geq 0 \\ 0 \leq \beta_1, \beta_2, \beta_3, \beta_4 \leq 1}} l(\theta_1, \beta_1, \beta_2, \beta_3, \beta_4) = \sup_{\substack{\theta_1 \geq 0 \\ 0 \leq \beta_1, \beta_3, \beta_4 \leq 1}} l(\theta_1, \beta_1, 1, \beta_3, \beta_4)$$

Case 2: Zero failure at two disjoint stress levels.

Without loss of generality let us assume $n_2 = 0$ and $n_4 = 0$. Therefore, the log-likelihood function is given by

$$\begin{aligned} l(\theta_1, \beta_1, \beta_2, \beta_3, \beta_4) &= -n \ln \theta_1 - (n_3 + n_5) \ln \beta_1 - (n_3 + n_5) \ln \beta_2 - n_5 \ln \beta_3 - n_5 \ln \beta_4 \\ &\quad - \frac{D_1}{\theta_1} - \frac{D_2}{\beta_1 \theta_1} - \frac{D_3}{\beta_1 \beta_2 \theta_1} - \frac{D_4}{\beta_1 \beta_2 \beta_3 \theta_1} - \frac{D_5}{\beta_1 \beta_2 \beta_3 \beta_4 \theta_1}. \end{aligned} \tag{15}$$

In this case also the function (15) has a unique global maximum and it does not have any local maximum. For fixed β_1 and β_3, the function (15) is maximized when

$$\widehat{\theta}_1(\beta_1, \beta_3) = \frac{\beta_1 D_1 + D_2}{n_1 \beta_1}, \quad \widehat{\beta}_2(\beta_1, \beta_3) = \frac{n_1(\beta_3 D_3 + D_4)}{n_3 \beta_3 (\beta_1 D_1 + D_2)}, \quad \widehat{\beta}_4(\beta_1, \beta_3) = \frac{n_3 D_5}{n_5(\beta_3 D_3 + D_4)}.$$

Therefore, the profile log-likelihood of β_1 and β_3 is given by

$$l_{13}(\beta_1, \beta_3) = n_1 \ln \beta_1 + n_3 \ln \beta_3 - n_1 \ln(\beta_1 D_1 + D_2) - n_3 \ln(\beta_3 D_3 + D_4).$$

Since, $l_{13}(\beta_1,\ \beta_3)$ can be expressed as the sum of two functions where one depends on β_1 only and other depends on β_2 only and

$$\frac{\delta l_{13}(\beta_1,\beta_3)}{\delta\beta_1} = \frac{n_1 D_2}{\beta_1(\beta_1 D_1 + D_2)} \geq 0 \quad \text{and} \quad \frac{\delta l_{13}(\beta_1,\beta_3)}{\delta\beta_3} = \frac{n_3 D_4}{\beta_3(\beta_3 D_3 + D_4)} \geq 0,$$

the profile log-likelihood of β_1 and β_3, is an increasing function of β_1 and β_3 $(0 \leq \beta_1, \beta_3 \leq 1)$. Therefore, $\widehat{\beta}_1 = 1$ and $\widehat{\beta}_3 = 1$ maximize the log-likelihood function. Hence,

$$\sup_{\substack{\theta_1 \geq 0 \\ 0\leq\beta_1,\beta_2,\beta_3,\beta_4\leq 1}} l(\theta_1,\beta_1,\beta_2,\beta_3,\beta_4) = \sup_{\substack{\theta_1 \geq 0 \\ 0\leq\beta_2,\beta_4\leq 1}} l(\theta_1,1,\beta_2,1,\beta_4).$$

Case 3: Zero failure at two consecutive stress levels.

Without loss of generality let us assume that $n_3 = 0$ and $n_4 = 0$. Therefore, the log-likelihood function without the additive constant is given by

$$\begin{aligned} l(\theta_1,\beta_1,\beta_2,\beta_3,\beta_4) &= -n\ln\theta_1 - (n_2+n_5)\ln\beta_1 - n_5\ln\beta_2 - n_5\ln\beta_3 - n_5\ln\beta_4 \\ &\quad -\frac{D_1}{\theta_1} - \frac{D_2}{\beta_1\theta_1} - \frac{D_3}{\beta_1\beta_2\theta_1} - \frac{D_4}{\beta_1\beta_2\beta_3\theta_1} - \frac{D_5}{\beta_1\beta_2\beta_3\beta_4\theta_1}. \end{aligned}$$

In this case also it can be shown as before that for

$$\sup_{\substack{\theta_1 \geq 0 \\ 0\leq\beta_1,\beta_2,\beta_3,\beta_4\leq 1}} l(\theta_1,\beta_1,\beta_2,\beta_3,\beta_4) = \sup_{\substack{\theta_1 \geq 0 \\ 0\leq\beta_1,\beta_4\leq 1}} l(\theta_1,\beta_1,1,1,\beta_4).$$

Hence, the result follows. In general it can be shown that the MLE of $\beta_k = 1$ if $n_{k+1} = 0$ for $k = 1, 2, \ldots m - 1$.

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