Order Restricted Bayesian Analysis of a Simple Step Stress Model

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Abstract

In this article we consider a simple step stress set up under the cumulative exposure model assumption. At each stress level the lifetime distribution of the experimental units are assumed to follow the generalized exponential distribution. We provide the order restricted Bayesian inference of the model parameters by considering the fact that the expected lifetime of the experimental units are larger in lower stress level. Analysis and the related results are extended to different censoring schemes also. The Bayes estimates and the associated credible intervals of the unknown parameters are constructed using importance sampling technique. We perform extensive simulation experiments both for the complete and censored samples to see the performances of the proposed estimators. We analyze two simulated and one real data sets for illustrative purposes. An optimal value of the stress changing time is obtained by minimizing the total posterior coefficient of variations of the unknown parameters.

Key Words Step-stress life-tests; Cumulative exposure model; Bayes estimate; Generalized Exponential distribution; Credible interval; Censoring scheme; Optimality.

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1 Introduction

Nowadays, since the products are highly reliable, it is very difficult to get sufficient failure time data in a normal condition during a reasonable experimental time. The accelerated life testing (ALT) procedures are proposed to overcome this problem. The ALT method has been introduced in a reliability experiment mainly to obtain more failures in a shorter interval of time. In an ALT experiment, units are put into a higher stress level than the usual that ensures early failure of the experimental units. Interested readers are referred to Nelson [13] and Bagdanavicius and Nikulin [3] for an exposure to different ALT models. The step stress life test (SSLT) model is a special type of the ALT model in which stress level can be changed during the experiment. In a conventional SSLT, the stress levels are changed at pre-fixed time points. Hence, in a conventional SSLT experiment, n experimental units are placed into life testing experiment at an initial stress level S_1 and then the stress level changes to $S_2, S_3, \ldots, S_m$ at prefixed time points $\tau_1, \tau_2, \ldots, \tau_{m-1}$, respectively. If $m = 2$, i.e., in case of only two stress levels, the experiment is known as the simple SSLT experiment.

The data collected from such an SSLT experiment, may then be extrapolated to estimate the underlying distribution of failure times under normal stress level. To connect the distributions of lifetime under different stress levels various models have been proposed in the literature. One such model was introduced by Seydyakin [15], and it is known as the cumulative exposure model (CEM). The CEM relates the distributions of lifetime under different stress levels by assuming that the residual life of the experimental units depends only on the cumulative exposure that the units have experienced, with no memory of how this exposure was accumulated. Latter this model was extensively studied by Nelson [13]. Interested readers are referred to a review article by Balakrishnan [4] or the recent monograph by Kundu and Ganguly [11], and the references cited therein.

In this paper we consider a simple step stress model when the lifetime distribution of experimental units follow generalized exponential (GE) distribution with the common shape parameter α but different scale parameters θ_1 and θ_2 at the two different stress levels. From now on it is assumed that a GE distribution with the shape parameter $\alpha > 0$ and scale parameter $\lambda > 0$, has the following probability density function (PDF)

$$f(t;\alpha,\lambda) = \alpha\lambda(1-e^{-\lambda t})^{\alpha-1}e^{-\lambda t}; \quad t \geq 0, \tag{1}$$

zero otherwise, and it will be denoted by GE(α, λ). The GE distribution was first considered by Gupta and Kundu [9] as an alternative to the well known gamma or Weibull distributions. It is also an extension of the exponential distribution, and it also can have increasing or decreasing hazard functions similar to the gamma and Weibull distributions. The GE distribution has a decreasing density function if the shape parameter is less than one and the density function becomes unimodal if the shape parameter is greater than one. This distribution has a very good interpretation in case of integer shape parameter. If the shape parameter is an integer, this distribution represents the lifetime of a parallel system where each component follows independent exponential distribution. It is observed, see for example Gupta and Kundu [10], that there are many cases where GE provides a better fit than the gamma or Weibull distribution. Interested readers are referred to the article by Nadarajah [12] for a survey on the GE distribution and the recent monograph by Al-Hussaini and Ahsanullah [2] for the development of the different exponentiated distributions. It may be mentioned that Abdel-Hamid and AL-Hussaini [1] considered the inference of the parameters of a GE distribution for simple SSLT model for Type-I censored data.

In a step stress model the basic assumption is that the expected lifetime of units under higher stress level is shorter than under the lower stress level. Therefore, this information can be incorporated by considering the order restriction on the scale parameters as $\theta_1 < \theta_2$. It seems although for a step-stress model, the order restricted inference is a natural choice, not much work has been done along this line mainly due to analytical difficulty. The order restricted inference for an exponential step stress model was first considered by Balakrishnan et al. [5] in case of Type-I and Type-II censored data. It is observed that for exponential model, although the maximum likelihood estimators (MLEs) of the unknown parameters can be obtained in explicit forms, the associated exact confidence intervals cannot be obtained in explicit form. Bayesian inference seems to be a reasonable choice in this case. Samanta et al. [14] developed the order restricted Bayesian inference for exponential simple step stress model. They obtained the Bayes estimates and the associated credible intervals of the unknown parameters under the squared error loss function based on importance sampling technique. The results have been developed for different censoring schemes also.

The main aim of this paper is to provide the Bayesian inference on order restricted parameters of a GE distribution for a simple SSLT model. It is assumed that at the two different stress levels the lifetime distributions of the items follow GE(α, θ_1) and GE(α, θ_2), respectively with $\theta_1 < \theta_2$. Moreover, it is assumed that it satisfies the CEM assumptions. We consider the Bayesian inference on the unknown parameters under a fairly flexible prior assumptions (the details of the priors will be provided in the next section). First we consider the complete sample, and provide the Bayes estimates and the associated credible intervals based on importance sampling technique. The necessary theoretical results for the convergence of the corresponding importance sampling procedure are also provided. The results are extended for different other censoring schemes, namely for Type I censoring, Type II censoring, Type I hybrid censoring scheme (HCS), introduced by Epstein [8], and for Type II hybrid censoring scheme, introduce by Childs et al. [6], also. Extensive Monte Carlo simulations are performed for complete and censored samples to see the performance of the proposed method, and they are quite satisfactory. Two simulated and one real data sets have been analyzed for illustrative purposes.

Finally we consider the 'optimal' simple SSLT model under the same assumptions. Similar to the idea proposed by Zhang and Meeker [16], we propose to choose the 'optimal' value of τ_1, so that the sum of the posterior coefficient of variations of α, θ_1 and θ_2 is minimum. Since the posterior coefficient of variations of the unknown parameters cannot be obtained in explicit forms, we use Lindley's approximation for the posterior coefficient of variations, and provide a methodology to choose the 'optimum' τ_1. A small table with the 'optimal' values of τ_1 is provided for different sample sizes and for different parameter values.

The rest of the paper is organized as follows. In Section 2, we provide the model and the necessary prior assumptions. The Bayesian inference of the unknown parameters for complete sample is provided in Section 3, and for different censoring schemes the results are provided in Section 4. Simulation and data analysis results are reported in Section 5. In Section 6 we consider the optimality of the simple SSLT model, and finally we conclude the paper in Section 7.

2 Model Assumption and Prior Information

Consider the simple step-stress model with two stress levels S_1 and S_2. Suppose n items are put into an experiment under the stress level S_1 and the stress level is changed to S_2 at a pre-fixed time τ_1. The failure times, denoted by $t_{1:n} < t_{2:n} < t_{3:n} < \ldots < t_{n:n}$, of the unit placed on the test are recorded chronologically. It is assumed that the lifetimes have a generalized exponential (GE) distribution under both the stress levels, with the common shape parameter α and different scale parameters, say θ_1 and θ_2 under stress level S_1 and S_2, respectively. It is further assumed that the lifetime satisfies CEM assumptions. Hence, the cumulative distribution function (CDF) of the lifetime is given by

$$F(t) = \begin{cases} (1-e^{-\theta_1 t})^{\alpha} & \text{if } 0 < t \leq \tau_1 \\ (1-e^{-\theta_2(t+\frac{\theta_1}{\theta_2}\tau_1-\tau_1)})^{\alpha} & \text{if } \tau_1 < t < \infty, \end{cases} \tag{2}$$

and corresponding PDF is given by

$$f(t) = \begin{cases} \alpha\theta_1(1-e^{-\theta_1 t})^{\alpha-1}e^{-\theta_1 t} & \text{if } 0 < t \leq \tau_1 \\ \alpha\theta_2(1-e^{-\theta_2(t+\frac{\theta_1}{\theta_2}\tau_1-\tau_1)})^{\alpha-1}e^{-\theta_2(t+\frac{\theta_1}{\theta_2}\tau_1-\tau_1)} & \text{if } \tau_1 < t < \infty. \end{cases} \tag{3}$$

The purpose of an ALT procedure is to increase the stress level which ensures the early failure of the experimental units. Hence, it is reasonable to assume that the mean lifetime at the stress level S_1 is larger than that at the stress level S_2, i.e.,

$$\frac{1}{\theta_2}[\psi(\alpha+1)-\psi(1)] < \frac{1}{\theta_1}[\psi(\alpha+1)-\psi(1)], \tag{4}$$

where $\psi(\cdot)$ is the digamma function. From (4), it follows that $\theta_1 < \theta_2$. We use this information in our prior assumption as follows. Let us assume $\theta_1 = \beta\theta_2$, where $0 < \beta < 1$. Suppose the prior belief of the experimenter is measured by the function $\pi(\alpha, \theta_2, \beta)$, which is given by

$$\pi(\alpha, \theta_2, \beta) = \pi_1(\alpha)\pi_2(\theta_2)\pi_3(\beta).$$

It is assumed that $\alpha \sim Gamma(a_0, b_0)$, $\theta_2 \sim Gamma(a_1, b_1)$, $\beta \sim Beta(a_2, b_2)$ and they are independently distributed. The joint prior distribution of $(\alpha, \theta_2, \beta)$, is given by

$$\pi(\alpha, \theta_2, \beta) \propto \beta^{a_2-1}(1-\beta)^{b_2-1}e^{-a_0\alpha}\alpha^{b_0-1}e^{-a_1\theta_2}\theta_2^{b_1-1}. \tag{5}$$

3 Posterior Analysis and Bayesian Inference

Based on the joint prior distribution (5), and under the CEM assumptions, the joint posterior distribution of α, θ_2 and β is given by

$$\begin{aligned} l(\beta, \theta_2, \alpha|Data) \propto\ & \beta^{n_1+a_2-1}(1-\beta)^{b_2-1}\theta_2^{n+b_1-1}e^{-A_1(\beta)\theta_2}\alpha^{n+b_0-1}e^{-A_2(\beta,\theta_2)\alpha} \\ & \times \prod_{i=1}^{n_1}(1-e^{-\beta\theta_2 t_{i:n}})^{-1} \prod_{i=n_1+1}^{n}(1-e^{-\theta_2(t_i-\tau_1+\tau_1\beta)})^{-1}, \end{aligned} \tag{6}$$

where n_1 denotes the number of failures till τ_1, and

$$\begin{aligned} A_1(\beta) &= a_1 + \beta\sum_{i=1}^{n_1} t_{i:n} + \sum_{i=n_1+1}^{n}(t_{i:n} - \tau_1 + \tau_1\beta), \\ A_2(\beta,\, \theta_2) &= a_0 - \sum_{i=1}^{n_1} log(1-e^{-\beta\theta_2 t_{i:n}}) - \sum_{i=n_1+1}^{n} log(1-e^{-\theta_2(t_{i:n}-\tau_1+\tau_1\beta)}). \end{aligned}$$

Therefore, the Bayes estimate of some parametric function of (β, θ_2, α), say $g(\beta,\, \theta_2,\, \alpha)$, under the squared error loss function is

$$\hat{g}_B(\beta, \theta_2, \alpha) = E_{\beta,\theta_2,\alpha|Data}\left(g(\beta, \theta_2, \alpha)\right)$$
$$= \frac{\int_0^1 \int_0^\infty \int_0^\infty g(\beta, \theta_2, \alpha) l(\beta, \theta_2, \alpha|Data) d\alpha d\theta_2 d\beta}{\int_0^1 \int_0^\infty \int_0^\infty l(\beta, \theta_2, \alpha|Data) d\alpha d\theta_2 d\beta}, \tag{7}$$

provided the expectation exists. In general (7) cannot be obtained in explicit form. One can use approximation procedure like Lindley's approximation or Tierney and Kadane's Method. However, the associated credible interval cannot be constructed using these techniques. Hence, we propose to use importance sampling technique to compute the Bayes estimates and the associated credible intervals of the unknown parameters. Note that posterior density of (β, θ_2, α) can be written as

$$l(\beta,\, \theta_2,\, \alpha \,|\, Data) \propto h(\beta,\, \theta_2,\, \alpha) l_1(\beta) l_2(\theta_2 \,|\, \beta) l_3(\alpha \,|\, \theta_2,\, \beta), \tag{8}$$

where

$$\begin{aligned}
h(\beta, \theta_2, \alpha) &= \beta^{n_1+a_2-1}(1-\beta)^{b_2-1}\,[A_1(\beta)]^{-(n+b_1)}\,[A_2(\beta,\, \theta_2)]^{-(n+b_0)} \\
&\quad \textstyle\prod_{i=1}^{n_1}(1-e^{-\beta\theta_2 t_i})^{-1}\prod_{i=n_1+1}^{n}(1-e^{-\theta_2(t_i-\tau_1+\tau_1\beta)})^{-1}, \\
l_1(\beta) &= 1 \quad \text{for } 0<\beta<1, \\
l_2(\theta_2|\beta) &= \tfrac{[A_1(\beta)]^{n+b_1}}{\Gamma(n+b_1)}\,\theta_2^{n+b_1-1}\,e^{-A_1(\beta)\,\theta_2} \quad \text{for } \theta_2>0, \\
l_3(\alpha|\theta_2, \beta) &= \tfrac{[A_2(\beta,\theta_2)]^{n+b_0}}{\Gamma(n+b_0)}\,\alpha^{n+b_0-1}\,e^{-A_2(\beta,\theta_2)\,\alpha} \quad \text{for } \alpha>0.
\end{aligned}$$

Using equation (8), following algorithm can be executed to compute the Bayes estimate and the associated credible interval of some parametric function $g(\beta,\, \theta_2,\, \alpha)$ of β, θ_2 and α, as given in (7).

Algorithm 1

Step 1. Generate β_1 from Uniform$(0,1)$, θ_{21} from Gamma$(n+b_1,\, A_1(\beta_1))$, and α_1 from Gamma$(n+b_0,\, A_2(\beta_1,\, \theta_{21}))$ distribution.

Step 2. Repeat Step 1, N times to obtain $(\beta_1, \theta_{21}, \alpha_1)$, ..., $(\beta_N, \theta_{2N}, \alpha_N)$, where β_i, θ_{2i} and α_i is the generation of β, θ_2 and α at i-th ($i = 1, \ldots, N$) replication respectively.

Step 3. Calculate $g_i = g(\beta_i, \theta_{2i}, \alpha_i)$ and $w_i = \dfrac{h(\beta_i, \theta_{2i}, \alpha_i)}{\sum_{j=1}^{N} h(\beta_j, \theta_{2j}, \alpha_j)}$ for $i = 1, \ldots, N$.

Step 4. The approximate value of (7) can be obtained as $\sum_{i=1}^{N} w_i g_i$.

Step 5. Rearrange (g_1, w_1), (g_2, w_2), ..., (g_N, w_N) as $(g_{(1)}, w_{(1)})$, $(g_{(2)}, w_{(2)})$, ..., $(g_{(N)}, w_{(N)})$ where $g_{(1)} \leq g_{(2)} \leq \ldots \leq g_{(N)}$. Note that $w_{(i)}$'s are not ordered, they are just associated with $g_{(i)}$'s.

Step 6. A $100(1-\gamma)\%$ credible interval for $g(\beta,\, \theta_2,\, \alpha)$ can be obtain as $(g_{j_1},\, g_{j_2})$, where j_1 and j_2 satisfy

$$j_1,\, j_2 \in \{1,\, 2,\, \ldots,\, N\}\,, \quad j_1 < j_2, \quad \sum_{i=j_1}^{j_2} w_{(i)} \leq 1 - \gamma < \sum_{i=j_1}^{j_2+1} w_{(i)}. \tag{9}$$

The $100(1-\gamma)\%$ HPD credible interval (CRI) of $g(\beta,\, \theta_2,\, \alpha)$ becomes $\left(g_{(j_1^*)},\, g_{(j_2^*)}\right)$, where $1 \leq j_1^* < j_2^* \leq N$ satisfy

$$\sum_{i=j_1^*}^{j_2^*} w_{(i)} \leq 1 - \gamma < \sum_{i=j_1^*}^{j_2^*+1} w_{(i)}, \quad \text{and} \quad g_{(j_2^*)} - g_{(j_1^*)} \leq g_{(j_2)} - g_{(j_1)},$$

for all j_1 and j_2 satisfying (9).

4 Different Censoring Schemes and Posterior Analysis

Due to the experimental time and budget restrictions, the experimenter often terminates the experiment before the last unit fails. This is known as censoring in the statistical terminology. In this section we discuss different censoring schemes and associated posterior analysis based on the same prior and model assumptions. Consider the following general notations for different censoring schemes. n_1^* = number of failure before τ_1; n_2^* = number of failure between τ_1 and τ^*; τ^* = termination time of the experiment; n^* = total number of failure before τ^*.

4.1 Type-I Censoring

In Type-I censoring scheme we stop the experiment at a prefix time, say τ_2 and the number of observations failed under stress level S_1 and S_2 are n_1 and n_2 respectively. In this case observed data are one of the forms

(a) $\{\tau_1 < t_{1:n} < ... < t_{n_2:n} < \tau_2\}$,

(b) $\{t_{1:n} < t_{2:n} < ... < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < ... < t_{n_1+n_2:n} < \tau_2\}$,

(c) $\{t_{1:n} < t_{2:n} < ... < t_{n_1:n} < \tau_1 < \tau_2\}$.

Under Type-I censoring scheme posterior distribution can be written as

$$l(\beta,\, \theta_2,\, \alpha \,|\, Data) \propto h_1(\beta,\, \theta_2,\, \alpha) l_1(\beta) l_2(\theta_2 \,|\, \beta) l_3(\alpha \,|\, \theta_2,\, \beta), \tag{10}$$

where

$$\begin{aligned}
h_1(\beta, \theta_2, \alpha) &= \beta^{n_1^*+a_2-1}(1-\beta)^{b_2-1}\,[A_1(\beta)]^{-(n^*+b_1)}\,[A_2(\beta,\,\theta_2)]^{-(n^*+b_0)}\,[A_3(\beta,\,\theta_2,\alpha)]^{(n-n^*)} \\
&\quad \textstyle\prod_{i=1}^{n_1^*}(1-e^{-\beta\theta_2 t_i})^{-1}\prod_{i=n_1^*+1}^{n^*}(1-e^{-\theta_2(t_i-\tau_1+\tau_1\beta)})^{-1}, \\
l_1(\beta) &= 1 \quad \text{for } 0<\beta<1, \\
l_2(\theta_2|\beta) &= \tfrac{[A_1(\beta)]^{n^*+b_1}}{\Gamma(n^*+b_1)}\,\theta_2^{n^*+b_1-1}\,e^{-A_1(\beta)\,\theta_2} \qquad \text{for } \theta_2>0, \\
l_3(\alpha|\theta_2,\beta) &= \tfrac{[A_2(\beta,\,\theta_2)]^{n^*+b_0}}{\Gamma(n^*+b_0)}\,\alpha^{n^*+b_0-1}\,e^{-A_2(\beta,\,\theta_2)\,\alpha} \qquad \text{for } \alpha>0, \\
A_1(\beta) &= a_1+\beta\textstyle\sum_{i=1}^{n_1^*} t_{i:n}+\sum_{i=n_1^*+1}^{n^*}(t_{i:n}-\tau_1+\tau_1\beta), \\
A_2(\beta,\,\theta_2) &= a_0-\textstyle\sum_{i=1}^{n_1^*} log(1-e^{-\beta\theta_2 t_{i:n}})-\sum_{i=n_1^*+1}^{n^*} log(1-e^{-\theta_2(t_{i:n}-\tau_1+\tau_1\beta)}), \\
A_3(\beta,\,\theta_2,\,\alpha) &= 1-\{1-e^{-\theta_2(\tau^*-\tau_1+\tau_1\beta)}\}^{\alpha}.
\end{aligned}$$

Here $\tau^* = \tau_2$ and in case (a) $n_1^* = 0$, $n_2^* = n_2$, in case (b) $n_1^* = n_1$, $n_2^* = n_2$, and in case (c) $n_1^* = n$, $n_2^* = 0$.

The Bayes estimate and the associated HPD credible interval of any parametric function of $(\beta, \theta_2, \alpha)$ can be obtain using the same algorithm as discussed in case of complete data.

4.2 Type-II Censoring

In this censoring scheme the life testing experiment is terminated when the rth (prefixed number) failure occurs, i.e, the total number of failure is fixed but the termination time of the experiment is random. Available data under this censoring scheme is one of the forms.

(a) $\{\tau_1 < t_{1:n} < ... < t_{r:n}\}$,

(b) $\{t_{1:n} < t_{2:n} < ... < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < ... < t_{r:n}\}, n_1 < r$,

(c) $\{t_{1:n} < t_{2:n} < ... < t_{r:n} < \tau_1 < \tau_2\}$.

Based on Type-II censored data, the posterior analysis is same as that of Type-I censoring scheme with $\tau^* = t_{r:n}$, $n^* = r$ and in case (a) $n_1^* = 0$, $n_2^* = r$; in case (b) $n_1^* = n_1$, $n_2^* = r - n_1$; in case (c) $n_1^* = r$, $n_2^* = 0$. All other expressions and the following analysis are same as the Type-I censoring scheme.

4.3 Type-I Hybrid Censoring

The termination time in Type-I HCS is $\tau^* = min\{t_{r:n}, \tau_2\}$. Let n_1 and n_2 be the number of failures under stress level S_1 and S_2, respectively. Available data under this censoring scheme is one of the forms

(a) $\{\tau_1 < t_{1:n} < ... < t_{r:n}\}$ if $t_{r:n} \leq \tau_2$,

(b) $\{t_{1:n} < t_{2:n} < ... < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < ... < t_{r:n}\}$ if $t_{r:n} < \tau_2, n_1 < r$,

(c) $\{t_{1:n} < t_{2:n} < ... < t_{r:n} < \tau_1 < \tau_2\}$ if $t_{r:n} < \tau_1$,

(d) $\{\tau_1 < t_{1:n} < ... < t_{n_2:n} < \tau_2\}$ if $t_{r:n} > \tau_2$,

(e) $\{t_{1:n} < ... < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < ... < t_{n_1+n_2:n} < \tau_2\}$ if $t_{r:n} > \tau_2, n_1 < r$,

(f) $\{t_{1:n} < ... < t_{n_1:n} < \tau_1 < \tau_2\}$ if $t_{r:n} > \tau_2$.

Based on Type-I Hybrid censored data, the posterior analysis is same as that of Type-I censoring scheme with, for case (a) $n_1^* = 0, n_2^* = r$, for case (b) $n_1^* = n_1, n_2^* = r - n_1$, for case (c) $n_1^* = r, n_2^* = 0$, for case (d) $n_1^* = 0, n_2^* = n_2$, for case (e) $n_1^* = n_1, n_2^* = n_2$, and for case (f) $n_1^* = n_1, n_2^* = 0$. All other expressions and the following analysis are same as the Type-I censoring scheme.

4.4 Type-II Hybrid Censoring

In Type-II HCS the experiment is terminated at $\tau^* = max\{t_{r:n}, \tau_2\}$. In this case the experimental time and the number of failures both are random but it ensures at least r failures from the experiment. Let n_1 and n_2 be the number of failures under stress level S_1 and S_2, respectively. Available data under this censoring scheme is one of the forms

(a) $\{\tau_1 < t_{1:n} < ... < t_{r:n}\}$ if $t_{r:n} \geq \tau_2$,

(b) $\{t_{1:n} < t_{2:n} < ... < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < ... < t_{r:n}\}$ if $t_{r:n} \geq \tau_2, n_1 < r$,

(c) $\{\tau_1 < t_{1:n} < ... < t_{n_2:n} < \tau_2\}$ if $t_{r:n} < \tau_2$,

(d) $\{t_{1:n} < ... < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < ... < t_{n_1+n_2:n} < \tau_2\}$ if $t_{r:n} < \tau_2, n_1 < r$,

(e) $\{t_{1:n} < ... < t_{n_1:n} < \tau_1 < \tau_2\}$ if $t_{r:n} < \tau_2$.

Based on the Type-II Hybrid censored data, the posterior analysis is same as that of the Type-I censoring scheme with, for case (a) $n_1^* = 0, n_2^* = r$, for case (b) $n_1^* = n_1, n_2^* = r - n_1$, for case (c) $n_1^* = 0, n_2^* = n_2$, for case (d) $n_1^* = n_1, n_2^* = n_2$, for case (e) $n_1^* = n_1, n_2^* = 0$. All other expressions and the following analysis are same as the Type-II censoring scheme.

5 Simulation and Data Analysis

5.1 Simulation

In this section first we perform some simulation experiments on complete data to evaluate the performances of proposed method. In this simulation study we consider almost non-informative priors on α, β and θ_2, i.e., $a_0 = 0.0001$, $b_0 = 0.0001$, $a_1 = 0.0001$, $b_1 = 0.0001$, $a_2 = 1$ and $b_2 = 1$ as suggested by Congdon [7]. Results are obtained on 5000 replications with $N = 15000$. The Bayes estimates and the associated mean square errors (MSEs) for different parameter values are obtained and they are presented in Tables 1, 2 and 3. As expected, the MSEs of Bayes estimates decrease as n increases. Also we provide the 95% symmetric and HPD CRI of the different parameters in Tables 4, 5 and 6. It has been observed that most of the cases average estimates (AE) are overestimated for all the parameters. Hence, we also consider the left sided CRIs in simulation study.

Table 1: AEs and MSEs of α, θ_1, and θ_2 based on 5000 simulations with $\alpha = 0.6$, $\theta_1 = 0.1$, and $\theta_2 = 0.2$ for different values of n and τ.

		α		θ_1		θ_2	
n	τ	AE	MSE	AE	MSE	AE	MSE
10	5	0.7598	0.2177	0.1285	0.0061	0.2811	0.0758
	7	0.7669	0.2552	0.1246	0.0051	0.3040	0.1416
	9	0.7639	0.2163	0.1201	0.0041	0.3437	0.5255
20	5	0.6772	0.0633	0.1180	0.0027	0.2315	0.0102
	7	0.6745	0.0554	0.1157	0.0024	0.2394	0.0155
	9	0.6711	0.0545	0.1144	0.0021	0.2531	0.0272
30	5	0.6544	0.0331	0.1155	0.0018	0.2218	0.0056
	7	0.6483	0.0316	0.1125	0.0014	0.2207	0.0059
	9	0.6522	0.0294	0.1115	0.0013	0.2306	0.0229
40	5	0.6491	0.0235	0.1151	0.0015	0.2172	0.0038
	7	0.6427	0.0201	0.1119	0.0012	0.2161	0.0045
	9	0.6421	0.0201	0.1113	0.0010	0.2217	0.0062
50	5	0.6424	0.0173	0.1137	0.0012	0.2123	0.0026
	7	0.6406	0.0162	0.1127	0.0010	0.2160	0.0034
	9	0.6380	0.0152	0.1114	0.0009	0.2184	0.0045

Table 2: AEs and MSEs of α, θ_1, and θ_2 based on 5000 simulations with $\alpha = 1.0$, $\theta_1 = 0.1$, and $\theta_2 = 0.2$ for different values of n and τ.

		α		θ_1		θ_2	
n	τ	AE	MSE	AE	MSE	AE	MSE
10	5	1.3876	0.9952	0.1245	0.0048	0.2438	0.0214
	7	1.3850	1.0298	0.1222	0.0042	0.2574	0.0417
	9	1.3498	0.8898	0.1183	0.0036	0.2710	0.1230
20	5	1.1687	0.2340	0.1148	0.0022	0.2152	0.0049
	7	1.1596	0.2240	0.1130	0.0019	0.2204	0.0065
	9	1.1377	0.2006	0.1098	0.0016	0.2250	0.0105
30	5	1.1179	0.1374	0.1125	0.0016	0.2084	0.0029
	7	1.1159	0.1330	0.1099	0.0013	0.2093	0.0031
	9	1.1149	0.1277	0.1090	0.0012	0.2126	0.0042
40	5	1.1024	0.0981	0.1117	0.0014	0.2059	0.0021
	7	1.0934	0.0890	0.1091	0.0010	0.2060	0.0022
	9	1.0778	0.0781	0.1068	0.0008	0.2070	0.0027
50	5	1.0864	0.0746	0.1108	0.0012	0.2043	0.0016
	7	1.0739	0.0653	0.1080	0.0009	0.2050	0.0018
	9	1.0676	0.0633	0.1067	0.0007	0.2052	0.0022

Table 3: AEs and MSEs of α, θ_1, and θ_2 based on 5000 simulations with $\alpha = 1.5$, $\theta_1 = 0.1$, and $\theta_2 = 0.2$ for different values of n and τ.

		α		θ_1		θ_2	
n	τ	AE	MSE	AE	MSE	AE	MSE
10	5	2.0073	1.6746	0.1180	0.0030	0.2228	0.0089
	7	2.0745	1.9250	0.1167	0.0030	0.2309	0.0113
	9	2.0925	2.1423	0.1142	0.0026	0.2395	0.0380
20	5	1.7279	0.4353	0.1100	0.0016	0.2081	0.0032
	7	1.7431	0.4939	0.1077	0.0014	0.2090	0.0036
	9	1.7316	0.4566	0.1080	0.0013	0.2168	0.0047
30	5	1.6468	0.2595	0.1052	0.0011	0.2020	0.0018
	7	1.6424	0.2727	0.1050	0.0010	0.2023	0.0019
	9	1.6461	0.2619	0.1057	0.0009	0.2065	0.0026
40	5	1.6035	0.1714	0.1048	0.0010	0.2003	0.0014
	7	1.5871	0.1629	0.1033	0.0008	0.1986	0.0014
	9	1.5937	0.1617	0.1029	0.0007	0.2014	0.0018
50	5	1.5662	0.1269	0.1027	0.0009	0.1980	0.0011
	7	1.5718	0.1320	0.1023	0.0007	0.1982	0.0011
	9	1.5667	0.1205	0.1022	0.0006	0.2000	0.0014

Table 4: CPs and ALs of 95% CRI for α, θ_1 and θ_2 based on 5000 simulations with $\alpha = 0.6$, $\theta_1 = 0.1$, and $\theta_2 = 0.2$ for different values of n and τ.

		α						θ_1						θ_2					
		Left CRI		Symmetric CRI		HPD CRI		Left CRI		Symmetric CRI		HPD CRI		Left CRI		Symmetric CRI		HPD CRI	
n	τ	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL
10	5	95.36	1.3426	96.04	1.3458	95.06	1.2343	97.66	0.2657	97.26	0.2728	96.40	0.2504	95.70	0.5171	94.94	0.5722	95.18	0.5090
	7	94.88	1.3446	95.54	1.3312	94.52	1.2265	97.48	0.2489	97.12	0.2497	95.86	0.2298	95.58	0.6246	95.36	0.7248	95.18	0.6178
	9	95.14	1.3254	95.42	1.3088	94.82	1.2037	96.96	0.2354	97.54	0.2331	95.82	0.2144	96.10	0.8630	96.20	1.0492	95.90	0.8578
20	5	95.82	0.9474	96.42	0.8134	95.84	0.7687	97.60	0.2114	96.66	0.1927	96.06	0.1813	95.48	0.2932	94.58	0.3044	94.10	0.2824
	7	96.18	0.9395	95.94	0.7905	96.16	0.7485	97.40	0.2003	95.58	0.1749	95.22	0.1646	95.30	0.3314	94.78	0.3510	94.84	0.3220
	9	96.00	0.9388	95.50	0.7772	95.78	0.7377	97.54	0.1954	95.36	0.1650	95.28	0.1550	95.86	0.4657	95.68	0.5419	95.56	0.4578
30	5	95.72	0.8218	95.64	0.6303	95.44	0.5992	97.68	0.1918	95.34	0.1590	95.00	0.1506	94.86	0.2311	93.64	0.2345	92.96	0.2192
	7	95.80	0.8124	95.28	0.6040	95.24	0.5758	97.22	0.1799	95.10	0.1406	94.36	0.1330	95.36	0.2476	95.14	0.2550	94.42	0.2371
	9	95.86	0.8162	95.34	0.5979	95.28	0.5701	97.08	0.1743	94.10	0.1310	93.80	0.1234	95.40	0.2860	95.80	0.3010	94.82	0.2769
40	5	96.26	0.7632	95.62	0.5341	95.68	0.5085	97.26	0.1809	94.20	0.1381	93.50	0.1310	94.88	0.1961	92.98	0.1952	92.10	0.1833
	7	96.22	0.7558	95.74	0.5111	95.78	0.4877	97.42	0.1699	93.26	0.1211	93.16	0.1147	95.06	0.2123	94.50	0.2145	93.66	0.2010
	9	96.16	0.7543	95.30	0.4993	95.08	0.4765	97.10	0.1647	92.52	0.1115	92.32	0.1050	94.78	0.2383	95.20	0.2454	93.66	0.2284
50	5	96.56	0.7210	95.56	0.4658	95.48	0.4432	97.62	0.1721	93.14	0.1225	92.70	0.1162	94.98	0.1707	92.82	0.1677	91.72	0.1578
	7	96.34	0.7211	95.10	0.4487	95.16	0.4276	97.88	0.1646	91.18	0.1084	91.16	0.1023	95.38	0.1922	94.26	0.1909	93.32	0.1793
	9	96.26	0.7207	95.28	0.4392	95.14	0.4184	97.36	0.1583	90.82	0.0983	90.62	0.0924	95.00	0.2098	94.90	0.2123	93.64	0.1981

Table 5: CPs and ALs of 95% CRI for α, θ_1 and θ_2 based on 5000 simulations with $\alpha = 1$, $\theta_1 = 0.1$, and $\theta_2 = 0.2$ for different values of n and τ.

		α						θ_1						θ_2					
		Left CRI		Symmetric CRI		HPD CRI		Left CRI		Symmetric CRI		HPD CRI		Left CRI		Symmetric CRI		HPD CRI	
n	τ	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL
10	5	96.08	2.7174	97.08	2.8248	95.30	2.5520	98.76	0.2508	98.54	0.2558	96.82	0.2368	95.68	0.3918	95.14	0.3921	93.82	0.3673
	7	95.70	2.6620	96.80	2.7352	94.58	2.4778	97.96	0.2359	97.66	0.2337	95.24	0.2178	96.38	0.4433	95.34	0.4586	95.20	0.4208
	9	95.06	2.5851	95.72	2.6204	93.48	2.3878	96.84	0.2218	96.94	0.2153	94.08	0.2010	96.12	0.6773	95.70	0.8150	95.44	0.6574
20	5	96.38	1.8108	97.34	1.6592	95.50	1.5566	98.40	0.2051	97.92	0.1907	96.08	0.1807	94.96	0.2664	95.18	0.2441	93.78	0.2339
	7	95.32	1.7860	96.12	1.5936	93.84	1.4995	97.40	0.1927	97.40	0.1708	94.68	0.1626	94.66	0.2880	95.02	0.2675	94.04	0.2558
	9	94.80	1.7307	95.80	1.5116	93.70	1.4269	97.28	0.1819	96.74	0.1558	95.02	0.1486	94.98	0.3135	95.46	0.2999	94.54	0.2844
30	5	95.96	1.5564	96.84	1.3035	94.76	1.2371	97.86	0.1883	97.68	0.1632	95.12	0.1557	94.18	0.2225	95.10	0.1930	93.32	0.1862
	7	95.84	1.5466	95.64	1.2462	93.74	1.1889	97.34	0.1756	97.28	0.1434	94.54	0.1373	94.86	0.2358	95.56	0.2070	94.80	0.1998
	9	95.90	1.5414	94.96	1.2094	93.16	1.1568	96.90	0.1684	96.10	0.1308	93.76	0.1255	94.54	0.2540	95.86	0.2285	94.62	0.2199
40	5	95.82	1.4405	96.26	1.1248	93.98	1.0746	97.56	0.1792	97.08	0.1467	93.98	0.1403	94.32	0.1985	94.66	0.1651	93.64	0.1597
	7	96.26	1.4271	96.06	1.0664	94.02	1.0233	97.16	0.1671	96.58	0.1280	93.70	0.1228	94.36	0.2107	95.52	0.1774	94.26	0.1721
	9	96.36	1.3962	96.14	1.0107	94.62	0.9721	97.30	0.1586	96.84	0.1149	94.56	0.1106	94.10	0.2242	95.62	0.1942	94.64	0.1883
50	5	96.10	1.3578	96.04	1.0029	94.30	0.9606	97.32	0.1724	96.72	0.1351	93.66	0.1293	94.12	0.1823	94.86	0.1467	93.96	0.1423
	7	96.30	1.3421	96.26	0.9439	94.54	0.9073	97.18	0.1607	96.50	0.1166	93.48	0.1121	94.42	0.1950	95.12	0.1588	94.14	0.1546
	9	95.88	1.3281	95.44	0.8982	93.56	0.8659	96.74	0.1534	95.90	0.1042	93.30	0.1004	93.96	0.2071	95.64	0.1738	94.22	0.1690

Table 6: CPs and ALs of 95% CRI for α, θ_1 and θ_2 based on 5000 simulations with $\alpha = 1.5$, $\theta_1 = 0.1$, and $\theta_2 = 0.2$ for different values of n and τ.

		α						θ_1						θ_2					
		Left CRI		Symmetric CRI		HPD CRI		Left CRI		Symmetric CRI		HPD CRI		Left CRI		Symmetric CRI		HPD CRI	
n	τ	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL
10	5	96.14	4.0881	97.82	4.2513	96.02	3.8536	99.10	0.2352	99.28	0.2377	98.14	0.2217	94.80	0.3361	95.02	0.3148	93.36	0.2992
	7	96.36	4.1680	97.86	4.3001	95.54	3.8968	98.62	0.2207	98.78	0.2161	96.52	0.2019	95.60	0.3597	95.72	0.3398	94.28	0.3215
	9	96.10	4.1478	97.00	4.2346	95.00	3.8512	98.08	0.2083	97.96	0.1984	95.40	0.1860	95.38	0.3998	95.62	0.3933	94.22	0.3629
20	5	96.04	2.7793	97.86	2.5945	95.84	2.4207	98.24	0.1944	98.90	0.1802	96.82	0.1701	94.04	0.2493	95.42	0.2049	93.90	0.1969
	7	96.62	2.7795	97.74	2.5297	95.82	2.3624	97.90	0.1799	98.30	0.1579	95.64	0.1491	94.18	0.2603	94.76	0.2145	93.40	0.2063
	9	96.10	2.7503	97.20	2.4378	94.50	2.2821	97.20	0.1730	97.70	0.1442	94.70	0.1363	94.40	0.2834	95.80	0.2392	94.70	0.2294
30	5	97.20	2.3413	98.10	2.0186	96.40	1.8986	98.60	0.1744	99.10	0.1528	97.60	0.1441	93.60	0.2133	94.40	0.1608	92.50	0.1545
	7	95.60	2.3496	97.20	1.9468	94.70	1.8403	97.60	0.1634	98.10	0.1325	95.10	0.1253	92.20	0.2220	94.50	0.1691	93.30	0.1628
	9	95.50	2.3318	96.50	1.8576	93.30	1.7544	98.00	0.1571	95.80	0.1184	93.50	0.1119	93.70	0.2369	94.50	0.1842	93.30	0.1774
40	5	96.80	2.1207	97.80	1.6984	95.90	1.6066	97.60	0.1647	98.10	0.1356	94.90	0.1280	90.50	0.1923	92.20	0.1355	90.50	0.1301
	7	96.50	2.1081	97.40	1.6193	94.60	1.5290	97.00	0.1536	97.40	0.1162	94.40	0.1098	92.40	0.2004	94.10	0.1427	93.30	0.1377
	9	95.46	2.1156	96.72	1.5526	94.06	1.4738	96.56	0.1467	96.80	0.1025	93.58	0.0968	92.40	0.2136	94.54	0.1563	92.90	0.1508
50	5	96.84	1.9516	97.96	1.4749	96.18	1.3982	97.14	0.1560	97.90	0.1228	95.28	0.1159	91.32	0.1780	92.76	0.1186	90.92	0.1138
	7	96.52	1.9784	97.20	1.4123	94.86	1.3377	97.06	0.1462	96.90	0.1039	93.78	0.0980	90.96	0.1873	93.28	0.1259	91.46	0.1212
	9	95.94	1.9781	96.38	1.3479	94.36	1.2799	96.46	0.1404	96.60	0.0913	93.24	0.0862	91.66	0.1991	93.78	0.1384	92.48	0.1335

We have further performed some simulation experiments based on Type-I and Type-II censored data. We have taken the same parameter values and the priors. The order restricted Bayes estimates and the associated MSEs of Type-I and Type-II censored data are presented in Tables 14 and 15, respectively. 95% CRIs of censored data are provided in Tables 16 and 17. Tables 14 to 17 are provided in the Appendix A.2. Censored data simulation results are very similar to that of complete data. In all the cases the parameter estimates are very consistent and the coverage percentages (CP) are very close to the nominal values. Also average lengths (AL) of CRIs are gradually decreases as sample size increases.

5.2 Data Analysis

5.2.1 Simulated Data Analysis

Here we consider the analysis of two simulated data sets; one the shape parameter is less than one and other it is greater than one. Data presented in Table 7 is generated from (2) with $\alpha = 0.6$, $\theta_1 = 0.1$ $\theta_2 = 0.2$, $n = 20$ and $\tau_1 = 5$. Artificially we have created Type-I and Type-II censored data by taking $\tau_2 = 8$ and $r = 16$, respectively. Prior assumptions are same as considered in simulation study. For Type-I censored data the Bayes estimates of α, θ_1, and θ_2 under the squared error loss function are $0.6995, 0.1032$, and 0.2747, respectively. In case of Type-II censored data Bayes estimates of α, θ_1, and θ_2 are $0.6244, 0.0840$ and 0.2659 respectively. Different CRIs for both Type-I and Type-II of censoring schemes are given in Table 8.

We analyze another data presented in Table 9 which is generated from the (2) with $\alpha = 1.5$. All other parameter values are same as the first data set. Here also we have considered Type-I and Type-II censored data. The Bayes estimates of α, θ_1, and θ_2 in Type-I censoring are $1.2787, 0.1109$, and 0.2269, respectively. In Type-II censored data Bayes estimates of α, θ_1, and θ_2 are $1.2147, 0.1041$, and 0.2220, respectively. 90%, 95% and 99% CRIs for both Type-I and Type-II censoring schemes are reported in Table 10.

Table 7: Type-I and Type-II censored data for analysis with $\alpha = 0.6$

Censoring Scheme	Stress Level	Data							
Type-I and Type-II	S_1	0.0185	0.0763	1.0137	1.2043	1.3411	1.3968	2.6797	3.4931
Type-I	S_2	5.1680	5.2476	5.4308	5.9575	7.2580	7.5416	7.7453	
Type-II	S_2	5.1680	5.2476	5.4308	5.9575	7.2580	7.5416	7.7453	8.0116

Table 8: CRIs for the unknown parameters for data in Table 7

		Type-I Censored data						Type-II Censored data					
		Left		Symmetric		HPD		Left		Symmetric		HPD	
Parameters	Level	LL	UL	LL	UL	LL	UL	LL	UL	LL	UL	LL	UL
α	90%	0.1186	0.9681	0.4207	1.1003	0.4129	1.0244	0.0852	0.8891	0.3713	1.0242	0.2727	0.8892
	95%	0.1186	1.1003	0.4050	1.1396	0.4152	1.1396	0.0852	1.0242	0.2738	1.1271	0.2727	1.0242
	99%	0.1186	1.2732	0.3015	1.3605	0.3015	1.2732	0.0852	1.2267	0.2738	1.2989	0.2727	1.2295
θ_1	90%	0.0001	0.1472	0.0593	0.1676	0.0561	0.1564	0.0001	0.1323	0.0375	0.1550	0.0375	0.1362
	95%	0.0001	0.1676	0.0561	0.1853	0.0484	0.1704	0.0001	0.1550	0.0375	0.1747	0.0375	0.1625
	99%	0.0001	0.2098	0.0484	0.2255	0.0466	0.2219	0.0001	0.1983	0.0267	0.2152	0.0203	0.1996
θ_2	90%	0.1041	0.3981	0.1469	0.4496	0.1234	0.4048	0.0976	0.3903	0.1397	0.4496	0.1352	0.4118
	95%	0.1041	0.4496	0.1252	0.4973	0.1163	0.4535	0.0976	0.4496	0.1354	0.5042	0.1031	0.4506
	99%	0.1041	0.5548	0.1113	0.5845	0.1041	0.5548	0.0976	0.5618	0.1034	0.5987	0.1031	0.5657

Table 9: Type-I and Type-II censored data for analysis with $\alpha = 1.5$

Censoring Scheme	Stress Level	Data						
Type-I and Type-II	S_1	0.6277	0.7266	2.2977	2.8450	3.0599	3.3134	
Type-I	S_2	5.1058	5.4453	5.5445	6.3469	7.1927	7.2401	7.5872
Type-II	S_2	5.1058	5.4453	5.5445	6.3469	7.1927	7.2401	7.5872
		8.0156	8.0383	10.7256				

Table 10: CRIs for the unknown parameters for data in Table 9

		Type-I Censored data						Type-II Censored data					
		Left		Symmetric		HPD		Left		Symmetric		HPD	
Parameters	Level	LL	UL	LL	UL	LL	UL	LL	UL	LL	UL	LL	UL
α	90%	0.1246	1.9342	0.6761	2.1827	0.6668	2.0429	0.1286	1.8628	0.6028	2.1363	0.4506	1.8835
	95%	0.1246	2.1827	0.6378	2.4342	0.5816	2.2610	0.1286	2.1363	0.4769	2.4069	0.4360	2.1514
	99%	0.1246	2.7845	0.5143	3.0832	0.4637	2.8129	0.1286	2.8159	0.4506	3.1361	0.4248	2.8789
θ_1	90%	0.0004	0.1805	0.0518	0.2056	0.0424	0.1830	0.0001	0.1776	0.0311	0.2037	0.0135	0.1777
	95%	0.0004	0.2056	0.0494	0.2297	0.0424	0.2115	0.0001	0.2037	0.0210	0.2264	0.0135	0.2038
	99%	0.0004	0.2572	0.0347	0.2748	0.0330	0.2579	0.0001	0.2565	0.0135	0.2789	0.0135	0.2574
θ_2	90%	0.0585	0.3322	0.1061	0.3784	0.1135	0.3816	0.0690	0.3192	0.1183	0.3523	0.1013	0.3287
	95%	0.0585	0.3784	0.0764	0.4248	0.0585	0.3784	0.0690	0.3523	0.1015	0.3819	0.1013	0.3731
	99%	0.0585	0.4746	0.0585	0.5132	0.0585	0.4746	0.0690	0.4209	0.0774	0.4490	0.0690	0.4209

5.2.2 Solar Lighting Device Data Set

A simple step stress test was conducted in order to asses the reliability characteristics of a solar lighting device. Thirty five (35) devices are put on a life test at the normal operating temperature 293K, and then the stress factor temperature is changed to 353K at the time point $\tau_1 = 5$ (in hundred hours). The experiment was terminated at the time point $\tau_2 = 6$ (in hundred hours). Thirty one (31) failures occur before τ_2 and among them fifteen (15) devices are failed at first stress and remaining sixteen (16) devices are failed at second stress level. The data are presented in Table 11.

Table 11: Solar lighting device dataset

Stress Level	Data										
S_1	0.140	0.783	1.324	1.582	1.716	1.794	1.883	2.293	2.660	2.674	2.725
	3.085	3.924	4.396	4.612	4.892						
S_2	5.002	5.022	5.082	5.112	5.147	5.238	5.244	5.247	5.305	5.337	5.407
	5.408	5.445	5.483	5.717							

We analyze the solar light data set based on the assumptions that at any stress level life time of devices follow GE distribution. We have obtained the order restricted Bayes estimates and different CRIs of model parameters. The order restricted Bayes estimates of α, θ_1 and θ_2 are respectively 1.4434, 0.1810 and 1.7921. CRIs of parameters are presented in Table 12.

Table 12: CRIs for the unknown parameters for data in Table 11

		Left		Symmetric		HPD	
Parameters	Level	LL	UL	LL	UL	LL	UL
α	90%	0.1948	2.0435	0.8623	2.2657	0.8249	2.1474
	95%	0.1948	2.2657	0.8149	2.4153	0.7491	2.3247
	99%	0.1948	2.6514	0.6694	2.8292	0.6200	2.6677
θ_1	90%	0.0003	0.2609	0.1028	0.2856	0.0982	0.2654
	95%	0.0003	0.2856	0.1009	0.3051	0.0989	0.2942
	99%	0.0003	0.3284	0.0797	0.3480	0.0797	0.3413
θ_2	90%	0.1357	2.4480	1.1284	2.6483	1.0702	2.5322
	95%	0.1357	2.6483	1.0273	2.8873	0.9308	2.7295
	99%	0.1357	3.1060	0.8437	3.2785	0.7681	3.1655

Now one natural question whether the GE distribution fits the data set or not. We have used the Kolmogorov-Smirnov (K-S) statistic, which quantifies the distance between the empirical distribution function of the data set and the cumulative distribution function of the fitted distribution function, for that purpose. The K-S distance and associated p-value is 0.2070 and 0.1212, respectively. It indicates that we cannot reject the null hypothesis at the 10% level of significance that the data are coming from a GE distribution.

6 Optimality of Test Plan

In the previous section we have obtained the Bayes estimates of the unknown parameters when the stress changing time τ_1 is pre-fixed. In this section we consider the problem of choosing the optimal value of τ_1, for a simple step-stress experiment. We obtain an optimal value of τ_1 by minimizing the sum of posterior coefficient of variations of α, θ_1 and θ_2. Since explicit form of the equation (7) cannot be obtained, we have used Lindley's approximation to calculate the posterior coefficient of variations of the unknown parameters. See Appendix A.1 for the detailed derivations of the Lindley's approximation. By minimizing sum of the posterior coefficient of variations, an optimal value of τ_1 can be obtained by using the following algorithm.

Algorithm 1

Step 1. For given $\alpha, \theta_1, \theta_2, \tau_1$ and n generate data from CEM.

Step 2. Obtain the posterior variance of all the parameters using Lindley's approximation as explained in Appendix A.1.

Step 3. Repeat Step 1 and Step 2, N times and take the average of variances.

Step 4. Calculate the coefficients of variation for Bayes estimates of $\alpha, \theta_1, \theta_2$.

Coefficient of Variation $= \frac{\text{posterior standard deviation}}{\text{posterior mean}}$

Step 5. Take the sum of coefficients of variation for Bayes estimates of α, θ_1 and θ_2.

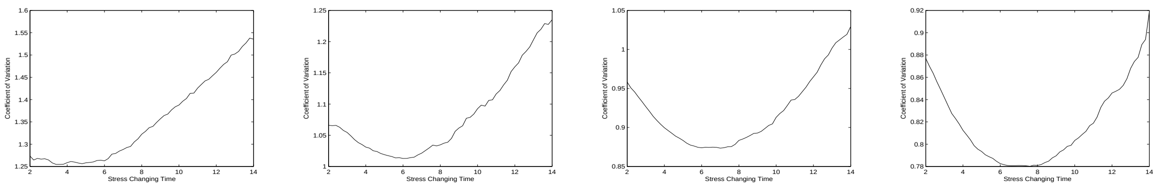
Step 6. Repeat Step 1 - Step 5 for different values of τ_1 within its range.

Step 7. Choose τ_1 for which the sum of coefficients of variation is minimum.

We have obtained numerically the optimal values of the stress changing times for different sample sizes and for different parameter values. It has been observed that the posterior variance of α is decreasing with the increase of τ_1. As expected the posterior variance of θ_1 has a decreasing trend and the posterior variance of θ_2 increases with τ_1. However, if we consider total dispersion of three parameters in terms of coefficient of variation, it is initially decreasing and then increasing as τ_1 increases. Hence, we have obtained a point where the total dispersion is minimum and which is the optimal value of the stress changing time τ_1. The experimental results and the plots of the sum of the coefficient of variations are given below.

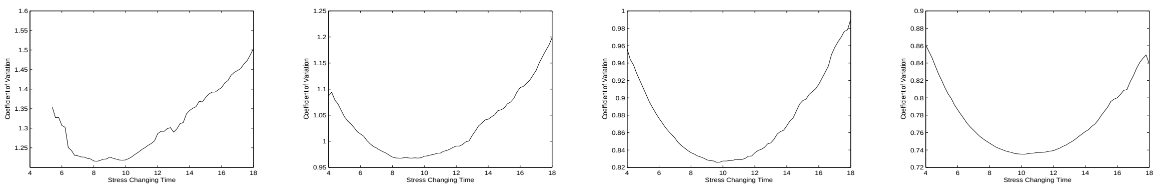
Table 13: Optimal value of τ_1 for different n and α with $\theta_1 = 0.1, \theta_2 = 0.2$

α	n	Optimal value of τ_1
0.6	20	3.6
0.6	30	6.4
0.6	40	7.4
0.6	50	7.2
1.0	20	8.4
1.0	30	8.2
1.0	40	9.4
1.0	50	10.0
1.5	20	10.8
1.5	30	13.0
1.5	40	13.4
1.5	50	13.4



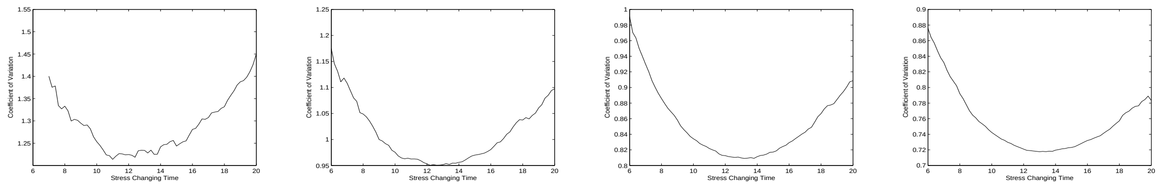
(a) $\alpha = 0.6, n = 20$ (b) $\alpha = 0.6, n = 30$ (c) $\alpha = 0.6, n = 40$ (d) $\alpha = 0.6, n = 50$

Figure 1: Plot of total coefficient of variation for different values of τ_1 with parameter values $\alpha = 0.6, \theta_1 = 0.1$ and $\theta_2 = 0.2$



(a) $\alpha = 1.0, n = 20$ (b) $\alpha = 1.0, n = 30$ (c) $\alpha = 1.0, n = 40$ (d) $\alpha = 1.0, n = 50$

Figure 2: Plot of total coefficient of variation for different values of τ_1 with parameter values $\alpha = 1.0, \theta_1 = 0.1$ and $\theta_2 = 0.2$



(a) $\alpha = 1.5, n = 20$ (b) $\alpha = 1.5, n = 30$ (c) $\alpha = 1.5, n = 40$ (d) $\alpha = 1.5, n = 50$

Figure 3: Plot of total coefficient of variation for different values of τ_1 with parameter values $\alpha = 1.5, \theta_1 = 0.1$ and $\theta_2 = 0.2$

7 Conclusion

In this paper we have considered the ordered restricted Bayesian inference of the unknown parameters of the GE distributions when the data are coming from a step-step model. It is assumed that the lifetime distribution satisfies the CEM assumptions. We have assumed a fairly flexible priors on the ordered parameters, and based on that we propose to use importance sampling technique to compute the Bayes estimates and the associated credible intervals. Extensive simulation experiments are performed for different sample sizes and for different parametric values. It is observed that the proposed method works quite well in practice. Finally we consider choosing the optimal value for the stress changing time. We choose the optimal value of optimal τ_1, so that the sum of the posterior coefficient of variations is minimum. Since the posterior coefficient of variations cannot be obtained in explicit forms, we suggest to use Lindley's approximation to compute the posterior coefficient of variations. A small table is provided for optimal values of τ_1, for different sample sizes and for different parametric values mainly for practical uses.

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A Appendix

A.1 Three Parameters Lindley's Approximation

For the three parameter case, using the notation $(\lambda_1, \lambda_2, \lambda_3) = (\alpha, \theta_2, \beta)$ Lindley's approximation of Bayes estimator of any function $g(\lambda_1, \lambda_2, \lambda_3)$ can be given by

$$\begin{aligned} E_{\lambda_1,\lambda_2,\lambda_3|Data}\left(g(\lambda_1, \lambda_2, \lambda_3)\right) &= g(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3) + \tfrac{1}{2}\textstyle\sum_{i=1}^{3}\sum_{j=1}^{3} u_{ij}\sigma_{ij} + \sum_{i=1}^{3}\sum_{j=1}^{3} u_i\rho_j\sigma_{ij} \\ &\quad + \tfrac{1}{2}\textstyle\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} L_{ijk}U_k\sigma_{ij}, \end{aligned} \tag{11}$$

where

$$L_{ijk} = \frac{\delta^3 L}{\delta\lambda_i \delta\lambda_j \delta\lambda_k}; \qquad i,j,k = 1(1)3 \quad \text{and L is log likelihood of the data;}$$
$$u_i = \frac{\delta g}{\delta\lambda_i}; \qquad i = 1(1)3;$$
$$u_{ij} = \frac{\delta^2 g}{\delta\lambda_i \delta\lambda_j}; \qquad i,j = 1(1)3;$$
$$\sigma_{ij} = (i,j)^{th} \text{ element of the inverse of the matrix having elements}((-L_{ij}));$$
$$\rho_i = \frac{\delta log\pi}{\delta\lambda_i};$$
$$U_k = \sum_{i=1}^{3} u_i \sigma_{ki}; \qquad k = 1(1)3,$$

$\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3$ are MLEs of $\lambda_1, \lambda_2, \lambda_3$ respectively and all of the quantities are evaluated at $(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3)$. The log likelihood function of the data under order restriction of parameter is given by

$$\begin{aligned} L(t_1, t_2, \ldots t_n|\alpha, \theta_2, \beta) =& log(n!) + nlog(\alpha) + n_1 log(\beta) + nlog(\theta_2) - \beta\theta_2 \sum_{k=1}^{n_1} t_k \\ &+ (\alpha-1)\sum_{k=1}^{n_1} log(1 - e^{-\beta\theta_2 t_k}) - \theta_2 \sum_{k=n_1+1}^{n} (t_k + \beta\tau_1 - \tau_1) \\ &+ (\alpha-1)\sum_{k=n_1+1}^{n} log(1 - e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}). \end{aligned} \tag{12}$$

The MLEs of α, θ_2, and β can be obtain by maximizing (12).

$$\frac{\delta L}{\delta\alpha} = 0 \Rightarrow \hat{\alpha} = \frac{-n}{\sum_{k=1}^{N_1} ln(1 - e^{-\beta\theta_2 t_k}) + \sum_{k=N_1+1}^{n} ln(1 - e^{-\theta_2(t_k+\beta\tau_1-\tau_1)})}, \tag{13}$$

$$\begin{aligned} \frac{\delta L}{\delta\theta_2} = 0 \Rightarrow \frac{n}{\theta_2} - \sum_{k=1}^{N_1} \beta t_k + (\alpha-1)\sum_{k=1}^{N_1} \frac{\beta t_k e^{-\beta\theta_2 t_k}}{1 - e^{-\beta\theta_2 t_k}} - \sum_{k=N_1+1}^{n} (t_k + \beta\tau_1 - \tau_1) \\ +(\alpha-1)\sum_{k=N_1+1}^{n} \frac{(t_k + \beta\tau_1 - \tau_1)e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}}{1 - e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}} = 0. \end{aligned} \tag{14}$$

For known $\beta(0 < \beta < 1)$, the estimate of θ_2 can be obtain by solving (14) numerically and hence an estimate of α from (13). The value of β between 0 and 1 and the corresponding estimates of α and θ_2, for which likelihood is maximum will be the MLEs of β, α and θ_2 respectively.

$$
\begin{aligned}
L_{11} &= \frac{\delta^2 l}{\delta \alpha^2} = -\frac{n}{\alpha^2}, \\
L_{12} &= L_{21} = \frac{\delta^2 l}{\delta \alpha \delta \theta_2} = \sum_{k=1}^{n_1} \frac{\beta t_k e^{-\beta\theta_2 t_k}}{1 - e^{-\beta\theta_2 t_k}} + \sum_{k=n_1+1}^{n} \frac{(t_k + \beta\tau_1 - \tau_1) e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}}{1 - e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}}, \\
L_{13} &= L_{31} = \frac{\delta^2 l}{\delta \alpha \delta \beta} = \sum_{k=1}^{n_1} \frac{\theta_2 t_k e^{-\beta\theta_2 t_k}}{1 - e^{-\beta\theta_2 t_k}} + \sum_{k=n_1+1}^{n} \frac{\theta_2 \tau_1 e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}}{1 - e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}}, \\
L_{22} &= \frac{\delta^2 l}{\delta \theta_2^2} = -\frac{n}{\theta_2^2} - (\alpha - 1) \sum_{k=1}^{n_1} \frac{\theta_2^2 t_k^2 e^{-\beta\theta_2 t_k}}{(1 - e^{-\beta\theta_2 t_k})^2} - (\alpha - 1) \sum_{k=n_1+1}^{n} \frac{(t_k + \beta\tau_1 - \tau_1)^2 e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}}{(1 - e^{-\theta_2(t_k+\beta\tau_1-\tau_1)})^2}, \\
L_{23} &= L_{32} = \frac{\delta^2 l}{\delta \theta_2 \delta \beta} = -\alpha \sum_{k=1}^{n_1} t_k - (\alpha - 1) \sum_{k=1}^{n_1} \frac{t_k(1 - e^{-\beta\theta_2 t_k}) - \beta\theta_2 t_k^2 e^{-\beta\theta_2 t_k}}{(1 - e^{-\beta\theta_2 t_k})^2} - \alpha(n - n_1)\tau_1 \\
&\quad -(\alpha - 1) \sum_{k=n_1+1}^{n} \frac{\tau_1(1 - e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}) - \theta_2\tau_1(t_k + \beta\tau_1 - \tau_1) e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}}{(1 - e^{-\theta_2(t_k+\beta\tau_1-\tau_1)})^2}, \\
L_{33} &= \frac{\delta^2 l}{\delta \beta^2} = -\frac{n_1}{\beta^2} - (\alpha - 1) \sum_{k=1}^{n_1} \frac{\theta_2^2 t_k^2 e^{-\beta\theta_2 t_k}}{(1 - e^{-\beta\theta_2 t_k})^2} - (\alpha - 1) \sum_{k=n_1+1}^{n} \frac{\theta_2^2 \tau_1^2 e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}}{(1 - e^{-\theta_2(t_k+\beta\tau_1-\tau_1)})^2},
\end{aligned}
$$

$$
\begin{aligned}
L_{111} &= \frac{\delta^3 L}{\delta\alpha^3} = \frac{2n}{\alpha^3}, \quad L_{112} = \frac{\delta^3 L}{\delta\alpha^2\delta\theta_2} = 0, \quad L_{113} = \frac{\delta^3 L}{\delta\alpha^2\delta\beta} = 0, \\
L_{122} &= \frac{\delta^3 L}{\delta\alpha\delta\theta_2^2} = -\sum_{k=1}^{n_1} \frac{\beta^2 t_k^2 e^{-\beta\theta_2 t_k}}{(1-e^{-\beta\theta_2 t_k})^2} - \sum_{k=n_1+1}^{n} \frac{(t_k+\beta\tau_1-\tau_1)^2 e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}}{(1-e^{-\theta_2(t_k+\beta\tau_1-\tau_1)})^2}, \\
L_{123} &= \frac{\delta^3 L}{\delta\alpha\delta\theta_2\delta\beta} = \sum_{k=1}^{n_1} [\frac{t_k(1-e^{-\beta\theta_2 t_k}) - \theta_2\beta t_k^2 e^{-\beta\theta_2 t_k}}{(1-e^{-\beta\theta_2 t_k})^2} - t_k] \\
&\quad + \sum_{k=n_1+1}^{n} [\frac{\tau_1(1-e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}) - \theta_2\tau_1(t_k+\beta\tau_1-\tau_1)e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}}{(1-e^{-\theta_2(t_k+\beta\tau_1-\tau_1)})^2} - \tau_1], \\
L_{133} &= \frac{\delta^3 L}{\delta\alpha\delta\beta^2} = -\sum_{k=1}^{n_1} \frac{\theta_2^2 t_k^2 e^{-\beta\theta_2 t_k}}{(1-e^{-\beta\theta_2 t_k})^2} - \sum_{k=n_1+1}^{n} \frac{\theta_2^2 \tau_1^2 e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}}{(1-e^{-\theta_2(t_k+\beta\tau_1-\tau_1)})^2}, \\
L_{222} &= \frac{\delta^3 L}{\delta\theta_2^3} = \frac{2n}{\theta_2^3} + (\alpha-1)\sum_{k=1}^{n_1} \frac{\beta^3 t_k^3 e^{-\beta\theta_2 t_k}(1+e^{-\beta\theta_2 t_k})}{(1-e^{-\beta\theta_2 t_k})^3} \\
&\quad +(\alpha-1)\sum_{k=n_1+1}^{n} \frac{(t_k+\beta\tau_1-\tau_1)^3 e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}(1+e^{-\theta_2(t_k+\beta\tau_1-\tau_1)})}{(1-e^{-\theta_2(t_k+\beta\tau_1-\tau_1)})^3}, \\
L_{223} &= \frac{\delta^3 L}{\delta\theta_2^2\delta\beta} = -(\alpha-1)\sum_{k=1}^{n_1} \frac{2\beta t_k^2 e^{-\beta\theta_2 t_k}}{(1-e^{-\beta\theta_2 t_k})^2} + (\alpha-1)\sum_{k=1}^{n_1} \frac{\beta^2\theta_2 t_k^3 e^{-\beta\theta_2 t_k}(1+e^{-\beta\theta_2 t_k})}{(1-e^{-\beta\theta_2 t_k})^3} \\
&\quad -(\alpha-1)\sum_{k=n_1+1}^{n} \frac{2\tau_1(t_k+\beta\tau_1-\tau_1)e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}}{(1-e^{-\theta_2(t_k+\beta\tau_1-\tau_1)})^2} \\
&\quad +(\alpha-1)\sum_{k=n_1+1}^{n} \frac{\theta_2\tau_1(t_k+\beta\tau_1-\tau_1)^2 e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}(1+e^{-\theta_2(t_k+\beta\tau_1-\tau_1)})}{(1-e^{-\theta_2(t_k+\beta\tau_1-\tau_1)})^3},
\end{aligned}
$$

$$
\begin{aligned}
L_{233} &= \frac{\delta^3 L}{\delta\theta_2\delta\beta^2} = -(\alpha-1)\sum_{k=1}^{n_1} \frac{2\theta_2 t_k^2 e^{-\beta\theta_2 t_k}}{(1-e^{-\beta\theta_2 t_k})^3} + (\alpha-1)\sum_{k=1}^{n_1} \frac{\beta\theta_2^2 t_k^3 e^{-\beta\theta_2 t_k}(1+e^{-\beta\theta_2 t_k})}{(1-e^{-\beta\theta_2 t_k})^3} \\
&\quad -(\alpha-1)\sum_{k=n_1+1}^{n} \frac{2\theta_2\tau_1^2 e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}}{(1-e^{-\theta_2(t_k+\beta\tau_1-\tau_1)})^3} \\
&\quad +(\alpha-1)\sum_{k=n_1+1}^{n} \frac{\theta_2^2\tau_1^2(t_k+\beta\tau_1-\tau_1)e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}(1+e^{-\theta_2(t_k+\beta\tau_1-\tau_1)})}{(1-e^{-\theta_2(t_k+\beta\tau_1-\tau_1)})^3}, \\
L_{333} &= \frac{\delta^3 L}{\delta\beta^3} = \frac{2n_1}{\beta^3} + (\alpha-1)\sum_{k=1}^{n_1} \frac{\theta_2^3 t_k^3 e^{-\beta\theta_2 t_k}(1+e^{-\beta\theta_2 t_k})}{(1-e^{-\beta\theta_2 t_k})^3} \\
&\quad +(\alpha-1)\sum_{k=n_1+1}^{n} \frac{\theta_2^3\tau_1^3 e^{-\theta_2(t_k+\beta\tau_1-\tau_1)}(1+e^{-\theta_2(t_k+\beta\tau_1-\tau_1)})}{(1-e^{-\theta_2(t_k+\beta\tau_1-\tau_1)})^3}.
\end{aligned}
$$

Note that L_{ijk} does not depends on the order of appearance of i, j and k.

$$\rho_1 = \frac{\delta log(\pi)}{\delta\alpha} = \frac{b_0 - 1}{\alpha} - a_0; \quad \rho_2 = \frac{\delta log(\pi)}{\delta\theta_2} = \frac{b_1 - 1}{\theta_2} - a_1 \quad \text{and} \quad \rho_3 = \frac{\delta log(\pi)}{\delta\beta} = \frac{a_2 - 1}{\beta} - \frac{b_2 - 1}{1 - \beta}$$

To obtain the posterior variance of the parameters we need to take below assumptions on the function $g(\alpha, \theta_2, \beta)$.

(a) To calculate posterior variance of α : $g = \alpha$ and $g = \alpha^2$.

(b) To calculate posterior variance of θ_1 : $g = \beta\theta_2$ and $g = \beta^2\theta_2^2$.

(c) To calculate posterior variance of θ_2 : $g = \theta_2$ and $g = \theta_2^2$.

In case (a) if $g = \alpha$, $u_1 = 1$, $u_2 = u_3 = 0$, $u_{ij} = 0, i, j = 1(1)3$.

If $g = \alpha^2$, $u_1 = 2\alpha$, $u_2 = u_3 = 0$, $u_{11} = 2$, $u_{ij} = 0$ for $i, j = 1(1)3$ and $(i, j) \neq (1, 1)$.

In case (b) if $g = \beta\theta_2$, $u_1 = 0$, $u_2 = \beta$, $u_3 = \theta_2$, $u_{23} = 1, u_{ij} = 0$, for $i, j = 1(1)3$ and $(i, j) \neq (2, 3)$.

If $g = \beta^2\theta_2^2$, $u_1 = 0$, $u_2 = 2\beta^2\theta_2$, $u_3 = 2\beta\theta_2^2$, $u_{11} = u_{12} = u_{13} = 0, u_{22} = 2\beta^2, u_{23} = 4\beta\theta_2, u_{33} = 2\theta_2^2$.

In case (c) if $g = \theta_2$, $u_2 = 1, u_1 = u_3 = 0$ $u_{ij} = 0, i, j = 1(1)3$.

If $g = \theta_2^2$, $u_2 = 2\theta_2$, $u_1 = u_3 = 0$, $u_{22} = 2$, $u_{ij} = 0$ for $i, j = 1(1)3$ and $(i, j) \neq (2, 2)$.

Note that $u_{ij} = u_{ji}$ for all $i, j = 1(1)3$.

Now posterior variance of the parameters can be obtain by using the equation (11).

A.2 Simulation Results

Table 14: AEs and MSEs of α, θ_1, and θ_2 based on 5000 simulations with $\alpha = 1.5$, $\theta_1 = 0.1$, and $\theta_2 = 0.2$ for different values of n, τ_1 and τ_2 of Type-I censored data.

			α		θ_1		θ_2	
n	τ_1	τ_2	AE	MSE	AE	MSE	AE	MSE
20	7	13	1.8329	0.8966	0.1114	0.0018	0.2079	0.0043
	9	13	1.8214	0.8455	0.1096	0.0016	0.2140	0.0062
	9	15	1.8132	0.7975	0.1080	0.0015	0.2103	0.0051
30	7	13	1.7538	0.4748	0.1111	0.0014	0.2052	0.0029
	9	13	1.7314	0.4446	0.1088	0.0011	0.2067	0.0038
	9	15	1.6914	0.3702	0.1071	0.0010	0.2059	0.0030
40	7	13	1.7024	0.3247	0.1106	0.0011	0.2040	0.0020
	9	13	1.6671	0.2831	0.1078	0.0008	0.2043	0.0028
	9	15	1.6541	0.2562	0.1064	0.0008	0.2004	0.0021
50	7	13	1.6716	0.2261	0.1105	0.0009	0.2017	0.0016
	9	13	1.6693	0.2228	0.1099	0.0007	0.2029	0.0022
	9	15	1.6395	0.1909	0.1075	0.0007	0.2011	0.0019

Table 15: AEs and MSEs of α, θ_1, and θ_2 based on 5000 simulations with $\alpha = 1.5$, $\theta_1 = 0.1$, and $\theta_2 = 0.2$ for different values of n, τ_1 and r of Type-II censored data.

			α		θ_1		θ_2	
n	τ_1	r	AE	MSE	AE	MSE	AE	MSE
20	7	15	1.9245	1.2123	0.1148	0.0022	0.2273	0.0107
	9	15	1.8758	0.9971	0.1115	0.0017	0.2406	0.0400
	9	17	1.8348	0.8609	0.1104	0.0017	0.2227	0.0081
30	7	23	1.7902	0.5586	0.1133	0.0016	0.2142	0.0040
	9	23	1.7353	0.4554	0.1096	0.0012	0.2220	0.0088
	9	27	1.7147	0.3786	0.1079	0.0011	0.2090	0.0033
40	7	32	1.7028	0.3272	0.1105	0.0012	0.2081	0.0026
	9	32	1.6905	0.3092	0.1079	0.0009	0.2083	0.0031
	9	36	1.6659	0.2506	0.1073	0.0009	0.2061	0.0023
50	7	42	1.6759	0.2408	0.1094	0.0010	0.2049	0.0018
	9	42	1.6440	0.2089	0.1073	0.0007	0.2049	0.0021
	9	45	1.6284	0.1846	0.1058	0.0007	0.2036	0.0018

Table 16: CPs and ALs of 95% CRI for α, θ_1 and θ_2 based on 5000 simulations with $\alpha = 1.5$, $\theta_1 = 0.1$, and $\theta_2 = 0.2$ for different values of n, τ_1 and τ_2 of Type-I censored data.

			α						θ_1						θ_2					
			Left CRI		Symmetric CRI		HPD CRI		Left CRI		Symmetric CRI		HPD CRI		Left CRI		Symmetric CRI		HPD CRI	
n	τ_1	τ_2	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL
20	7	13	95.64	3.1864	97.12	3.0328	94.28	2.7679	97.78	0.1947	98.14	0.1743	95.78	0.1629	94.24	0.2726	95.38	0.2530	93.22	0.2408
	9	13	95.44	3.1317	96.28	2.9208	93.46	2.6716	96.80	0.1840	97.00	0.1561	93.88	0.1455	95.08	0.3055	96.02	0.2982	94.20	0.2800
	9	15	95.16	3.0708	96.30	2.8365	93.02	2.6166	96.64	0.1803	97.22	0.1541	93.68	0.1445	94.70	0.2871	95.92	0.2680	93.96	0.2547
30	7	13	96.40	2.7229	97.18	2.3466	93.80	2.1774	98.04	0.1802	97.44	0.1466	94.94	0.1375	94.02	0.2326	95.40	0.2043	92.98	0.1961
	9	13	95.63	2.6491	96.20	2.2070	92.80	2.0526	97.23	0.1688	96.57	0.1271	94.13	0.1187	94.40	0.2556	95.70	0.2392	93.63	0.2268
	9	15	95.53	2.5628	96.00	2.1399	93.47	2.0030	96.83	0.1670	97.37	0.1293	94.47	0.1219	94.47	0.2461	95.77	0.2192	94.13	0.2103
40	7	13	96.47	2.4529	96.37	1.9535	93.23	1.8261	97.73	0.1709	97.20	0.1283	94.63	0.1207	94.67	0.2092	95.53	0.1772	94.00	0.1705
	9	13	95.30	2.3766	95.70	1.8153	92.60	1.6960	97.17	0.1595	96.67	0.1088	94.50	0.1015	93.90	0.2285	95.50	0.2077	92.93	0.1978
	9	15	96.10	2.3392	96.20	1.7972	93.67	1.6899	96.87	0.1584	96.93	0.1123	93.93	0.1058	93.57	0.2171	95.97	0.1867	93.53	0.1800
50	7	13	96.80	2.2919	96.63	1.7071	93.57	1.5995	97.83	0.1650	97.13	0.1159	94.80	0.1089	94.17	0.1911	95.13	0.1573	93.40	0.1517
	9	13	96.37	2.2664	95.47	1.6042	92.33	1.5057	98.13	0.1562	97.07	0.0979	95.27	0.0914	94.30	0.2102	95.37	0.1871	93.73	0.1784
	9	15	96.23	2.2150	96.23	1.5880	93.33	1.4986	97.23	0.1544	96.77	0.1019	93.50	0.0962	94.20	0.2030	95.33	0.1693	93.67	0.1636

Table 17: CPs and ALs of 95% CRI for α, θ_1 and θ_2 based on 5000 simulations with $\alpha = 1.5$, $\theta_1 = 0.1$, and $\theta_2 = 0.2$ for different values of n, τ_1 and r of Type-II censored data.

			α						θ_1						θ_2					
			Left CRI		Symmetric CRI		HPD CRI		Left CRI		Symmetric CRI		HPD CRI		Left CRI		Symmetric CRI		HPD CRI	
n	τ_1	r	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL
20	7	15	96.30	3.3992	96.98	3.2367	94.28	2.9648	97.80	0.2018	97.64	0.1819	95.74	0.1700	95.24	0.3174	94.98	0.3037	93.94	0.2866
	9	15	95.76	3.2467	96.10	3.0317	93.20	2.7877	97.40	0.1878	97.26	0.1617	95.46	0.1513	95.66	0.3764	96.16	0.3808	95.40	0.3513
	9	17	95.50	3.0823	96.16	2.8396	93.02	2.6304	97.48	0.1838	97.06	0.1584	94.32	0.1496	95.40	0.3124	95.64	0.2924	94.96	0.2782
30	7	23	96.28	2.7953	96.82	2.4164	93.64	2.2468	98.02	0.1845	97.14	0.1509	95.16	0.1419	94.80	0.2508	95.32	0.2245	94.02	0.2154
	9	23	96.38	2.6635	96.06	2.2360	93.52	2.0851	97.44	0.1715	96.58	0.1321	95.02	0.1241	95.42	0.2862	96.06	0.2708	95.28	0.2558
	9	27	96.08	2.5628	96.16	2.1224	93.24	2.0007	96.88	0.1671	96.58	0.1307	93.10	0.1243	94.78	0.2478	95.66	0.2105	94.46	0.2035
40	7	32	96.54	2.4596	96.36	1.9713	93.48	1.8483	97.62	0.1721	97.06	0.1319	94.98	0.1244	94.46	0.2163	94.88	0.1826	93.76	0.1764
	9	32	96.46	2.4068	95.78	1.8552	92.76	1.7464	97.06	0.1610	96.60	0.1148	93.88	0.1083	94.44	0.2333	95.70	0.2050	94.48	0.1974
	9	36	96.26	2.3261	96.50	1.7818	93.58	1.6901	97.28	0.1591	96.44	0.1157	93.58	0.1101	94.42	0.2233	95.16	0.1810	94.40	0.1758
50	7	42	96.46	2.2933	96.38	1.7292	93.18	1.6310	97.42	0.1650	96.80	0.1208	93.92	0.1144	94.32	0.1965	95.08	0.1553	93.80	0.1508
	9	42	96.22	2.2202	95.74	1.6018	92.68	1.5161	97.14	0.1549	96.30	0.1045	93.24	0.0990	94.52	0.2101	95.48	0.1735	94.74	0.1684
	9	45	96.48	2.1732	96.44	1.5640	93.70	1.4855	97.32	0.1525	96.48	0.1046	94.00	0.0995	93.66	0.2065	95.18	0.1610	94.38	0.1569