Order Restricted Bayesian Inference for Exponential Simple Step-stress Model

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Abstract

Step-stress model has received a considerable amount of attention in recent years. In the usual step-stress experiment, stress level is allowed to increase at each step to get rapid failure of the experimental units. The expected lifetime of the experimental unit is shortened as the stress level increases. Although, extensive amount of work has been done on step-stress models, not enough attention has been paid to analyze step-stress models incorporating this information. We consider a simple step-stress model and provide Bayesian inference of the unknown parameters under cumulative exposure model assumption. It is assumed that lifetime of the experimental units are exponentially distributed with different scale parameters at different stress levels. It is further assumed that the stress level increases at each step, hence the expected lifetime decreases. We try to incorporate this restriction using the prior assumptions. It is observed that different censoring schemes can be incorporated very easily under a general set up. Monte Carlo simulations have been performed to see the effectiveness of the proposed method, and two data sets have been analyzed for illustrative purposes.

Key Words and Phrases: Step-stress life-tests; cumulative exposure model; Type-I and Type-II censoring schemes; hybrid censoring scheme; progressive censoring scheme; prior distribution; posterior analysis; maximum likelihood estimator.

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1 Introduction

In many reliability experiments often an investigator has to wait a long period of time to observe failures. Accelerated Life Testing (ALT) experiment has been used quite often to observe early failures. In this set up, the experimental units are exposed to higher stress levels than usual to reduce the time to failure, hence to observe more failures within an affordable time. Data, collected in this method, needs to extrapolate to get back to the normal condition.

A particular case of the accelerated life-tests is step-stress life-test (SSLT), where the experimenter is allowed to change the stress levels during the life-testing experiment. In this case, a number of experimental units, say n, are placed on a test at an initial stress level s_1 and then the stress levels are changed to s_2, s_3, $\ldots$, s_{m+1} at some prefixed times, say at $\tau_1 < \tau_2 < \ldots < \tau_m$, respectively. A simple SSLT is a special case of SSLT, where only two stress levels are considered, and the stress level is changed from s_1 to s_2 at a prefixed time τ_1. Moreover, to analyze such data we need a model that relates the distributions of lifetimes under different stress levels to that of lifetimes under step-stress pattern. One such model is cumulative exposure model (CEM), first introduced by Seydyakin [17], further studied by several authors, see for example Bagdonavicius [1] and Nelson [16]. This model has been extensively discussed in the literature.

In this paper we consider a simple step-stress model, and it is assumed that the lifetime distributions at two different stress levels follow exponential distribution with mean lifetimes λ_1^{-1} and λ_2^{-1} respectively, where $\lambda_1 < \lambda_2$. Moreover, CEM is assumed. It may be mentioned that although, extensive amount of work has been done on step-stress models, not much attention has been paid to develop the order restricted inference. Balakrishnan et al. [3] considered the order restricted inference for exponential step-stress models when the data are Type-I or Type-II censored. They have mainly adopted the frequentist approach, and the maximum likelihood estimators (MLEs) of the unknown parameters are obtained using isotonic regression. It is observed that obtaining the exact joint distribution of the MLEs is not very easy, hence they derived the asymptotic distribution of the MLEs. Based on the asymptotic distribution, the asymptotic confidence intervals (CI) of the unknown parameters can be constructed. It is not immediate that how this method can be extended for other censoring schemes like hybrid and progressive censoring schemes.

In life-testing experiment often the data are censored. The most popular censoring schemes are Type-I and Type-II censoring schemes. Hybrid censoring scheme (HCS) is a mixture of the Type-I and Type-II censoring schemes, and it was introduced by Epstein [11]. From now on, the HCS proposed by Epstein [11] will be called as Type-I HCS. Similar to conventional Type-I censoring scheme, the main disadvantage of Type-I HCS is that almost all inferential results are based on the assumption that there are at least one failure. Moreover, there may be very few failures, hence the efficiency of the estimator might be very low. For these reasons Childs et al. [9] introduced Type-II HCS, which guarantees a minimum number of failures during the experiment. For an extensive survey of different hybrid censoring schemes, the readers are referred to Balakrishnan and Kundu [4]. Recently, progressive censoring scheme (PCS) has received significant attention in the statistical literature. The main advantage of progressive censoring schemes is that it is possible to remove experimental units during the experiment, even if they do not fail. For an exhaustive survey on PCSs, the readers are referred to the review article by Balakrishnan [2]. A brief review of the different censoring schemes is provided in Section 2.

Simple step-stress models under different censoring schemes are extensively studied based on the assumption that the lifetime of the experimental units follow exponential distributions with different scale parameters at different stress levels. From now on we call them as exponential step-stress models. Simple exponential step-stress model under Type-I censoring is considered by Balakrishnan et al. [8]. Balakrishnan et al. [5] considered simple exponential step-stress model under the Type-II censoring scheme. Simple exponential step-stress models under HCS-I and HCS-II are considered by Balakrishnan and Xie [7] and Balakrishnan and Xie [6], respectively. In all these cases the exact distributions of the unknown parameters are obtained, and they can be used to construct exact CIs. However it is observed that the exact distribution and hence the construction of associated CI is quite complicated in all these cases. Moreover, all the inferential issues are obtained without the order restriction on the unknown parameters. It is clear that the ordered restricted inference will be quite complicated in the frequentist set up, see Balakrishnan et al. [3]. It seems Bayesian analysis is a natural choice in these cases. Some work has been done on the Bayesian inference of the step-stress model, see for example Dorp et al. [10], Lee and Pan [13], Leu and Shen [14], Fan et al. [12], and Liu [15]. However, none of them dealt with the ordered restricted inference.

The main aim of this paper is to consider the order restricted Bayesian inference of the unknown parameters of a simple exponential step-stress model under different censoring schemes. We have assumed fairly flexible priors on the unknown parameters. It is observed that in all the cases the Bayes estimates of the unknown parameters cannot be obtained in explicit form. We propose to use importance sampling technique to compute Bayes estimate (BE) and also to construct associated credible interval (CRI). We also discuss construction of credible set for model parameters. Extensive Monte Carlo simulations are performed to see the effectiveness of the proposed method, and the performances are quite satisfactory. The analyses of two data sets have been performed for illustrative purposes.

Rest of the paper is organized as follows. In Section 2, we briefly discuss different censoring schemes and available data. Model assumptions and the prior information of the unknown parameters are considered in Section 3. In Section 4, maximum likelihood estimation of model parameters is briefly discussed under the order restriction for Type-I censored data. In Section 5, we provide the posterior analysis for different censoring schemes under the order restriction. Monte Carlo simulation results are presented in Section 6.1. Data analyses have been provided in Section 6.2. Finally, we conclude the article in Section 7.

2 Different Censoring Schemes and Available Data

A total of n units is placed on a simple SSLT experiment. The stress level is changed from s_1 to s_2 at a prefixed time τ_1, and $\tau_2 > \tau_1$ is another prefixed time. The positive integer $r \leq n$ is also pre-fixed. The role of r and τ_2 will be clear later. Let the ordered lifetimes of the items be denoted by $t_{1:n} < \ldots < t_{n:n}$. Now we briefly describe different censoring schemes, and available data in each case.

Type-I Censoring Scheme

The test is terminated when the time τ_2 on the test has been reached. For Type-I censoring the available data is of the form

(a) $\{\tau_1 < t_{1:n} < \ldots < t_{n_2:n} < \tau_2\}$,

(b) $\{t_{1:n} < \ldots < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < \ldots < t_{n_1+n_2:n} < \tau_2\}$,

(c) $\{t_{1:n} < \ldots < t_{n_1:n} < \tau_1 < \tau_2\}$.

Here, n_1 and n_2 are the number of failures at stress levels s_1 and s_2, respectively.

Type-II Censoring Scheme

The test is terminated when the rth failure takes place, *i.e.*, it is terminated at a random time $t_{r:n}$. In this case the available data is of the form

(a) $\{\tau_1 < t_{1:n} < \ldots < t_{r:n}\}$,

(b) $\{t_{1:n} < \ldots < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < \ldots < t_{r:n}\}$, $n_1 < r$,

(c) $\{t_{1:n} < \ldots < t_{r:n} < \tau_1 < \tau_2\}$.

In Case (b), n_1 is the number of failures at the stress level s_1.

Type-I Hybrid Censoring Scheme

In this case, the test is terminated at a random time $\tau^* = \min\{t_{r:n}, \tau_2\}$. For Type-I hybrid censoring scheme (HCS), the available data is of the form

(a) $\{\tau_1 < t_{1:n} < \ldots < t_{r:n}\}$ if $t_{r:n} < \tau_2$,

(b) $\{t_{1:n} < \ldots < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < \ldots < t_{r:n}\}$ if $t_{r:n} < \tau_2$, $n_1 < r$,

(c) $\{t_{1:n} < \ldots < t_{r:n} < \tau_1\}$ if $t_{r:n} < \tau_2$,

(d) $\{\tau_1 < t_{1:n} < \ldots < t_{n_2:n} < \tau_2\}$ if $t_{r:n} > \tau_2$,

(e) $\{t_{1:n} < \ldots < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < \ldots < t_{n_1+n_2:n} < \tau_2\}$ if $t_{r:n} > \tau_2$, $n_1 < r$,

(f) $\{t_{1:n} < \ldots < t_{n_1:n} < \tau_1 < \tau_2\}$ if $t_{r:n} > \tau_2$.

In Cases (b), (d), (e), and (f), n_1 and n_2 are the number of failures at stress levels s_1 and s_2, respectively.

Type-II Hybrid Censoring Scheme

In Type-II HCS, the experiment is terminated at a random time $\tau^* = \max\{t_{r:n}, \tau_2\}$. In this case the available data is of the form

(a) $\{\tau_1 < t_{1:n} < \ldots < t_{r:n}\}$ if $t_{r:n} \geq \tau_2$,

(b) $\{t_{1:n} < \ldots < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < \ldots < t_{r:n}\}$ if $t_{r:n} \geq \tau_2$, $n_1 < r$,

(c) $\{\tau_1 < t_{1:n} < \ldots < t_{n_2:n} < \tau_2\}$ if $t_{r:n} < \tau_2$,

(d) $\{t_{1:n} < \ldots < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < \ldots < t_{n_1+n_2:n} < \tau_2\}$ if $t_{r:n} < \tau_2$,

(e) $\{t_{1:n} < \ldots < t_{n_1:n} < \tau_1 < \tau_2\}$ if $t_{r:n} < \tau_2$.

In Cases (b), (c), (d), and (e), n_1 and n_2 are the number of failures at stress levels s_1 and s_2, respectively.

Type-II Progressive Censoring Scheme

In this case it is assumed that n experimental units are put in a life test. $R_1, \ldots, R_m$ are m prefixed non-negative integers such that

$$m + \sum_{j=1}^{m} R_j = n.$$

At the time of the first failure, say $t_{1:n}$, R_1 units are chosen at random from the remaining $(n-1)$ units and they are removed from the experiment. Similarly, at the time of the second failure, say $t_{2:n}$, R_2 units are chosen at random from the remaining $(n - R_1 - 2)$ surviving units and they are removed from the test, and so on. Finally at the time of the mth failure, say $t_{m:n}$, the rest of the $n - m - \sum_{j=1}^{m-1} R_j = R_m$ units are removed and the experiment is stopped. In this case the available data is of the form

(a) $\{\tau_1 < t_{1:n} < \ldots < t_{m:n}\}$,

(b) $\{t_{1:n} < \ldots < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < \ldots < t_{m:n}\}$,

(c) $\{t_{1:n} < \ldots < t_{m:n} < \tau_1\}$.

In Case (b), n_1 is number of failures at the stress level s_1.

3 Model Assumption and Prior Information

We consider a simple SSLT, where n identical units are placed on a life testing experiment at the initial stress level s_1. The stress level is increased to a higher level s_2 at a prefixed time τ_1. It is assumed that the lifetimes of the experimental units are independently and exponentially distributed random variables with different scale parameters at different stress levels. Probability density function (PDF) and the cumulative distribution function (CDF) of the lifetime under stress level s_i for $i = 1, 2$, is given by

$$f(t; \lambda_i) = \lambda_i \, e^{-\lambda_i t} \qquad \text{for} \quad 0 < t < \infty, \lambda_i > 0 \tag{1}$$

and

$$F(t; \lambda_i) = 1 - e^{-\lambda_i t} \qquad \text{for} \quad 0 < t < \infty, \lambda_i > 0, \tag{2}$$

respectively. Let us assume that the stress level is changed from s_1 to s_2 at the time point τ_1. It is further assumed that the failure time data comes from a CEM, hence, it has the following CDF;

$$G(t;\, \lambda_1,\, \lambda_2) = \begin{cases} F(t; \lambda_1) & \text{if} \quad 0 < t \le \tau_1 \\ F\left(t - \left(1 - \frac{\lambda_1}{\lambda_2}\right) \tau_1; \lambda_2\right) & \text{if} \quad \tau_1 < t < \infty. \end{cases} \tag{3}$$

The corresponding PDF is given by

$$g(t;\, \lambda_1,\, \lambda_2) = \begin{cases} \lambda_1 e^{-\lambda_1 t} & \text{if} \quad 0 < t \le \tau_1 \\ \\ \lambda_2 e^{-\lambda_2 (t + \frac{\lambda_1}{\lambda_2}\tau_1 - \tau_1)} & \text{if} \quad \tau_1 < t < \infty. \end{cases} \tag{4}$$

For developing the Bayesian inference, we need to assume some priors on the unknown parameters. We want the prior assumption on λ_1 and λ_2, so that it maintains the order restriction, namely, $\lambda_1 < \lambda_2$. We take the following priors on λ_1 and λ_2. It is assumed that λ_2 has a Gamma(a, b) distribution with $a > 0$ and $b > 0$, *i.e.*, it has the following PDF

$$\pi_1(\lambda_2) = \frac{b^a}{\Gamma(a)} \lambda_2^{a-1} e^{-b\lambda_2} \quad \text{for} \quad \lambda_2 > 0. \tag{5}$$

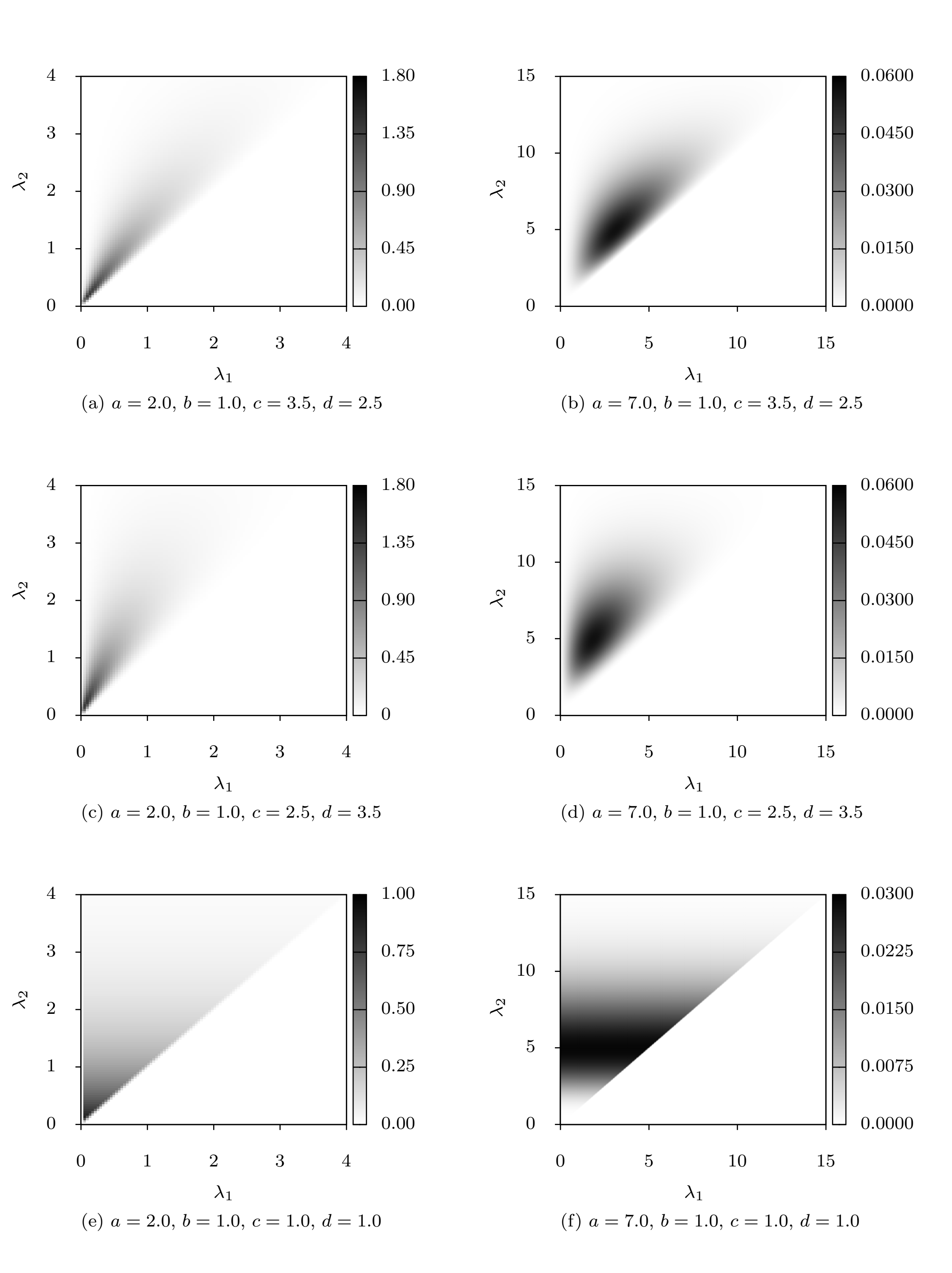
Moreover, $\lambda_1 = \alpha\, \lambda_2$ and α has a beta distribution with parameters $c > 0$ and $d > 0$, *i.e.*, the PDF of α is given by

$$\pi_2(\alpha) = \frac{1}{B(c, d)} \alpha^{c-1} (1 - \alpha)^{d-1} \quad \text{for} \quad 0 < \alpha < 1, \tag{6}$$

and the distribution of α is independent of λ_2. Therefore, the joint prior of (λ_1, λ_2) can be written as

$$\pi(\lambda_1, \lambda_2) = \frac{b^a}{\Gamma(a) B(c, d)} \lambda_2^{a-c-d} e^{-b\lambda_2} \lambda_1^{c-1} (\lambda_2 - \lambda_1)^{d-1} \quad \text{for} \quad 0 < \lambda_1 < \lambda_2 < \infty. \tag{7}$$

As the joint prior on $(\lambda_1,\, \lambda_2)$ is little complicated, a gray-scale plot is provided in Figure 1 for different values of hyper-parameters. In the plot black color represents the maximum value of density function, whereas white color represents the minimum value (which is zero) of density function. We have taken $b = 1.0$ only, as different values of b only effects the spread of the density function keeping the shape fixed.



(a) $a = 2.0$, $b = 1.0$, $c = 3.5$, $d = 2.5$

(b) $a = 7.0$, $b = 1.0$, $c = 3.5$, $d = 2.5$

(c) $a = 2.0$, $b = 1.0$, $c = 2.5$, $d = 3.5$

(d) $a = 7.0$, $b = 1.0$, $c = 2.5$, $d = 3.5$

(e) $a = 2.0$, $b = 1.0$, $c = 1.0$, $d = 1.0$

(f) $a = 7.0$, $b = 1.0$, $c = 1.0$, $d = 1.0$

Figure 1: Plot of prior density for different values of hyper-parameters.

4 Maximum Likelihood Estimator under Type-I Censoring Scheme

In this section we present maximum likelihood estimation of the scale parameters under the restriction $\lambda_1 \leq \lambda_2$, when data are Type-I censored. Let n_1^* and n_2^* denote the number of failures before the time τ_1 and between τ_1 and τ_2, respectively. They can be zero also. Let τ^* and n^* denote the termination time of the experiment and total number of failures observed before τ^*, respectively. Note that τ^* and n^* depend on the censoring scheme. In case of Type-I censoring scheme, $\tau^* = \tau_2$. For Case (a): $n_1^* = 0$, $n_2^* = n_2 > 0$, Case (b): $n_1^* = n_1 > 0$, $n_2^* = n_2 > 0$, Case (c): $n_1^* = n_1 > 0$, $n_2^* = 0$. In all the cases $n^* = n_1^* + n_2^*$. Based on the observations from a simple SSLT under Type-I censoring scheme, the likelihood can be written as

$$l_1(\lambda_1, \lambda_2 \,|\, \text{Data}) \quad \propto \quad \lambda_1^{n_1^*}\, \lambda_2^{n_2^*}\, e^{-\lambda_1 d_1 - \lambda_2 d_2}, \tag{8}$$

where $d_1 = \sum_{j=1}^{n_1^*} t_{j:n} + (n - n_1^*)\tau_1$, $d_2 = \sum_{j=n_1^*+1}^{n^*} (t_{j:n} - \tau_1) + (n - n^*)(\tau^* - \tau_1)$. Note that d_1 and d_2 are the total time elapsed by all the units at stress level s_1 and s_2, respectively. The unrestricted MLEs of λ_1 and λ_2 are given by

$$\widehat{\lambda}_1^* = \frac{n_1^*}{d_1} \qquad \text{and} \qquad \widehat{\lambda}_2^* = \frac{n_2^*}{d_2}.$$

Clearly, if $\widehat{\lambda}_1^* \leq \widehat{\lambda}_2^*$, MLEs of the scale parameters under the restriction $\lambda_1 \leq \lambda_2$ are given by

$$\widehat{\lambda}_1 = \widehat{\lambda}_1^* = \frac{n_1^*}{d_1} \qquad \text{and} \qquad \widehat{\lambda}_1 = \widehat{\lambda}_1^* = \frac{n_2^*}{d_2}.$$

As $l_1(\lambda_1, \lambda_2 \,|\, \text{Data})$ is unimodal function, if $\widehat{\lambda}_1^* > \widehat{\lambda}_2^*$, maximization of $l_1(\lambda_1, \lambda_2 \,|\, \text{Data})$ under the order restriction $\lambda_1 \leq \lambda_2$ is equivalent to maximization of $l_1(\lambda_1, \lambda_2 \,|\, \text{Data})$ under $\lambda_1 = \lambda_2$, and hence, in this case the MLEs of the scale parameters under the restriction $\lambda_1 \leq \lambda_2$ are given by

$$\widehat{\lambda}_1 = \widehat{\lambda}_2 = \frac{n_1^* + n_2^*}{d_1 + d_2}.$$

5 Posterior Analysis under Different Censoring Schemes

5.1 Type-I Censoring Scheme

Based on the likelihood function in (8), priors $\pi_1(\cdot)$ and $\pi_2(\cdot)$ mentioned in Section 3, posterior density function of $(\alpha,\, \lambda_2)$ becomes

$$l_2(\alpha, \lambda_2 \,|\, \text{Data}) \;\propto \alpha^{n_1^*+c-1}(1-\alpha)^{d-1}\lambda_2^{n^*+a-1}\, e^{-\lambda_2(d_1\alpha+d_2+b)} \qquad \text{if } 0<\alpha<1,\, \lambda_2>0. \tag{9}$$

The right hand side of (9) is integrable if $n_1^* + c > 0$ and $n^* + a > 0$. Bayes estimate of some function of α and λ_2, say $g(\alpha,\, \lambda_2)$, with respect to the squared error loss function, is posterior expectation of $g(\alpha,\, \lambda_2)$, *i.e.*,

$$\widehat{g}(\alpha,\, \lambda_2) = \int_0^1\int_0^\infty g(\alpha,\, \lambda_2) l_2(\alpha, \lambda_2 \,|\, \text{Data})\mathrm{d}\lambda_2\mathrm{d}\alpha. \tag{10}$$

Unfortunately, the close form of (10) cannot be obtained in most of the cases. One may use numerical techniques to compute (10). Alternatively, other approximations, like Lindey's approximation, can be used to compute (10). However, CRI for a parametric function cannot be constructed by these numerical methods. Hence we propose to use importance sampling to compute BE as well as to construct CRI of a parametric function in this article. Note that for $0 < \alpha < 1$ and $\lambda_2 > 0$, $l_2(\alpha,\, \lambda_2 \,|\, \text{Data})$ can be expressed as

$$l_2(\alpha,\, \lambda_2 \,|\, \text{Data}) = l_3(\alpha \,|\, \text{Data}) \times l_4(\lambda_2 \,|\, \alpha,\, \text{Data}), \tag{11}$$

where

$$l_3(\alpha \,|\, \text{Data}) = c_1 \frac{\alpha^{n_1^*+c-1}(1-\alpha)^{d-1}}{(d_1\alpha+d_2+b)^{a+n^*}}, \tag{12}$$

and

$$l_4(\lambda_2 \,|\, \alpha,\, \text{Data}) = \frac{\{d_1\alpha+d_2+b\}^{a+n^*}}{\Gamma(a+n^*)}\, \lambda_2^{a+n^*-1}\, e^{-\lambda_2(d_1\alpha+d_2+b)}. \tag{13}$$

The proportionality constant, c_1, for the posterior distribution of α given in (12) can be found using numerical techniques. However, generation from (12) is not a trivial issue. Hence, we propose to use the importance sampling (see Algorithm 5.1) to compute the BE and as well as to construct CRI of $g(\alpha, \lambda_2)$ noting the following representation of $l_2(\alpha,\, \lambda_2 \,|\, \text{Data})$. For

$0 < \alpha < 1$ and $\lambda_2 > 0$

$$l_2(\alpha,\, \lambda_2 \,|\, \text{Data}) = c_1 w_1(\alpha) \times l_4(\lambda_2 \,|\, \alpha,\, \text{Data}), \tag{14}$$

where

$$w_1(\alpha) = \frac{\alpha^{n_1^*+c-1}(1-\alpha)^{d-1}}{(d_1\alpha + d_2 + b)^{a+n^*}}. \tag{15}$$

Algorithm 5.1

Step 1. Generate α_1 from U(0, 1) distribution.

Step 2. For the given α_1, generate λ_{21} from (13).

Step 3. Continue the process M times to get $\{(\alpha_1,\, \lambda_{21}),\, \ldots,\, (\alpha_M,\, \lambda_{2M})\}$.

Step 4. Compute $g_i = g(\alpha_i,\, \lambda_{2i})$; $i = 1,\, 2,\, \ldots,\, M$.

Step 5. Calculate the weights $w_{1i} = \dfrac{c_1 w_1(\alpha_i)}{M}$; $i = 1,\, 2,\, \ldots,\, M$.

Step 6. Compute the BE of $g(\alpha, \lambda_2)$ as

$$\widehat{g}(\alpha, \lambda_2) = \sum_{j=1}^{M} w_{1j} g_j.$$

Step 7. To construct a $100(1-\gamma)\%$, $0 < \gamma < 1$, CRI of $g(\alpha,\, \lambda_2)$, first order g_j for $j = 1,\, \ldots,\, M$, say $g_{(1)} < g_{(2)} < \ldots < g_{(M)}$, and order w_{1j} accordingly to get $w_{1(1)}$, $w_{1(2)}$, $\ldots$, $w_{1(M)}$. Note that $w_{1(1)}$, $w_{1(2)}$, $\ldots$, $w_{1(M)}$ may not be ordered. A $100(1-\gamma)\%$ CRI can be obtained as $(g_{(j_1)}, g_{(j_2)})$, where j_1 and j_2 satisfy

$$j_1,\, j_2 \in \{1,\, 2,\, \ldots,\, M\}, \qquad j_1 < j_2, \qquad \sum_{i=j_1}^{j_2} w_{1(i)} \le 1-\gamma < \sum_{i=j_1}^{j_2+1} w_{1(i)}. \tag{16}$$

The $100(1-\gamma)\%$ height posterior density (HPD) CRI of $g(\alpha,\, \lambda_2)$ becomes $(g_{(j_1^*)},\, g_{(j_2^*)})$, where $j_1^* < j_2^*$ satisfy

$$j_1^*,\, j_2^* \in \{1,\, 2,\, \ldots,\, M\}, \quad \sum_{i=j_1^*}^{j_2^*} w_{1(i)} \le 1-\gamma < \sum_{i=j_1^*}^{j_2^*+1} w_{1(i)}, \quad g_{(j_2^*)} - g_{(j_1^*)} \le g_{(j_2)} - g_{(j_1)},$$

for all j_1 and j_2 satisfying (16).

Next we consider HPD credible set for $(\lambda_1,\, \lambda_2)$. Note that a subset $\mathcal{C}_\gamma$ of $\mathbb{R}^2$ is said to be a $100(1-\gamma)\%$, $0<\gamma<1$, HPD credible set for $(\lambda_1,\, \lambda_2)$ if

$$\mathcal{C}_\gamma = \left\{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : \lambda_1 = \alpha\lambda_2,\, l(\alpha,\, \lambda_2 \,|\, \text{Data}) \geq c_\gamma\right\},$$

where c_γ is such that

$$\iint_{\mathcal{C}_\gamma} l(\alpha,\, \lambda_2 \,|\, \text{Data}) d\lambda_2\, d\alpha = \gamma \iint_{\mathbb{R}^2} l(\alpha,\, \lambda_2 \,|\, \text{Data}) d\lambda_2\, d\alpha.$$

However, close form of the set $\mathcal{C}$ cannot be obtained, as the integration of the function $l(\alpha,\, \lambda_2 \,|\, \text{Data})$ is not possible analytically. Hence, we suggest the following algorithm to construct $100(1-\gamma)\%$ HPD credible set for $(\lambda_1,\, \lambda_2)$.

Algorithm 5.2

Step 1. Follow the first 5 steps of Algorithm 5.1.

Step 2. Arrange $\{(\alpha_1,\, \lambda_{21},\, w_{11}),\, \ldots,\, (\alpha_M,\, \lambda_{2M},\, w_{1M})\}$ according to the descending magnitude of the function $l(\alpha,\, \lambda_2 \,|\, \text{Data})$ at those points to get $\{(\widetilde{\alpha}_1,\, \widetilde{\lambda}_{21},\, \widetilde{w}_1),\, \ldots,\, (\widetilde{\alpha}_M,\, \widetilde{\lambda}_{2M},\, \widetilde{w}_M)\}$.

Step 3. Find the integer M_γ such that

$$\sum_{i=1}^{M_\gamma} \widetilde{w}_i \leq \gamma < \sum_{i=1}^{M_\gamma+1} \widetilde{w}_i.$$

Step 4. Construct the HPD credible set for $(\alpha,\, \lambda_2)$ as

$$\mathcal{C} = \left\{(\alpha,\, \lambda_2) : l(\alpha,\, \lambda_2 \,|\, \text{Data}) \geq l(\widetilde{\alpha}_{M_\gamma},\, \widetilde{\lambda}_{2M_\gamma} \,|\, \text{Data})\right\}.$$

Similar methodology can be applied for other censoring schemes, and we briefly mention all the cases in the subsequent subsections for completeness purposes.

5.2 Type-II Censoring Scheme

Based on the observed sample, the likelihood function is given in (8), where $\tau^* = t_{r:n}$, in Case (a), $n_1^* = 0$, $n_2^* = r$, in Case (b), $n_1^* = n_1$, $n_2^* = r - n_1$ and in Case (c), $n_1^* = r$, $n_2^* = 0$. d_1 and d_2 have the same expression as given in case of Type-I censoring.

5.3 Type-I Hybrid Censoring Scheme

Based on the data from Type-I HCS, the likelihood function is same as (8), where in Case (a), $n_1^* = 0$, $n_2^* = r$, in Case (b), $n_1^* = n_1$, $n_2^* = r - n_1$, in Case (c), $n_1^* = r$, $n_2^* = 0$, in Case (d), $n_1^* = 0$, $n_2^* = n_2$, in Case (e), $n_1^* = n_1$, $n_2^* = n_2$, and in Case (f), $n_1^* = n_1$, $n_2^* = 0$. Also in the Cases (a)-(c), $\tau^* = t_{r:n}$, where for the rest of the cases $\tau^* = \tau_2$. d_1 and d_2 have the same expression as given in case of Type-I censoring.

5.4 Type-II Hybrid Censoring Scheme

Based on the observed sample from Type-II HCS, the likelihood function is given in (8), where in Case (a), $n_1^* = 0$, $n_2^* = r$, for Case (b), $n_1^* = n_1$, $n_2^* = r - n_1$, in Case (c), $n_1^* = 0$, $n_2^* = n_2$, for Case (d), $n_1^* = n_1$, $n_2^* = n_2$ and for Case (e), $n_1^* = n_1$, $n_2^* = 0$. $\tau^* = t_{r:n}$ for Cases (a) and (b), where for the rest of the cases $\tau^* = \tau_2$. d_1 and d_2 have the same expression as given in case of Type-I censoring.

5.5 Progressive Type-II Censoring Scheme

With the observed Progressive Type-II censoring data, the likelihood function is given by (8), where for Case (a), $n_1^* = 0$, $n_2^* = m$, for Case (b), $n_1^* = n_1$, $n_2^* = m - n_1$ and for Case (c) $n_1^* = m$, $n_2^* = 0$. For all the cases $\tau^* = t_{m:n}$, $d_1 = \sum_{k=1}^{N_1^*}(R_k + 1)t_{k:n} + (n - N_1^* - \sum_{k=1}^{N^*} R_k)\tau_1$ and $d_2 = \sum_{k=N_1^*+1}^{m}(R_k + 1)(t_{k:n} - \tau_1)$.

In all the above cases, likelihood function are in the same form as Type-I censoring scheme and hence, the posterior density is also in the same form as given in (9). In all these cases computation of the BE and construction of the associated CRI for some function of α and λ_2 can be done exactly along the same line. One can also construct credible set for (λ_1, λ_2) following the same methodology.

Table 1: AE and MSE of MLE and BE of λ_1 and λ_2 based on 5000 simulations with $\lambda_1 = 0.083$, $\lambda_2 = 0.222$, $a = 0.001$, $b = 0.001$, $c = 1$, and $d = 1$ for the Type-I censored case.

			λ_1				λ_2			
			BE		MLE		BE		MLE	
n	τ_1	τ_2	AE	MSE	AE	MSE	AE	MSE	AE	MSE
10	5	6	0.092	0.0019	0.085	0.0021	0.275	0.0307	0.330	0.0956
		8	0.094	0.0018	0.088	0.0021	0.235	0.0143	0.254	0.0248
		10	0.096	0.0018	0.089	0.0021	0.238	0.0142	0.257	0.0234
10	7	8	0.088	0.0015	0.082	0.0016	0.284	0.0336	0.359	0.0954
		10	0.092	0.0016	0.087	0.0018	0.244	0.0248	0.274	0.0918
		12	0.093	0.0015	0.087	0.0018	0.246	0.0192	0.272	0.0360
10	9	10	0.087	0.0013	0.081	0.0013	0.306	0.0512	0.414	0.1583
		12	0.089	0.0013	0.085	0.0014	0.254	0.0281	0.296	0.1077
		14	0.092	0.0013	0.087	0.0014	0.251	0.0280	0.287	0.0850
20	5	6	0.088	0.0009	0.084	0.0010	0.225	0.0097	0.246	0.0184
		8	0.091	0.0009	0.086	0.0010	0.221	0.0059	0.236	0.0087
		10	0.091	0.0009	0.085	0.0011	0.221	0.0050	0.233	0.0069
20	7	8	0.087	0.0008	0.084	0.0008	0.236	0.0114	0.261	0.0240
		10	0.089	0.0008	0.085	0.0008	0.221	0.0076	0.238	0.0120
		12	0.090	0.0008	0.086	0.0008	0.225	0.0066	0.241	0.0097
20	9	10	0.086	0.0006	0.083	0.0007	0.243	0.0146	0.276	0.0323
		12	0.087	0.0006	0.085	0.0007	0.224	0.0084	0.244	0.0143
		14	0.090	0.0006	0.086	0.0007	0.227	0.0099	0.245	0.0170
30	5	6	0.087	0.0006	0.084	0.0007	0.219	0.0076	0.236	0.0128
		8	0.088	0.0006	0.084	0.0007	0.217	0.0046	0.230	0.0060
		10	0.090	0.0007	0.085	0.0007	0.221	0.0036	0.233	0.0045
30	7	8	0.086	0.0005	0.084	0.0006	0.219	0.0080	0.239	0.0147
		10	0.089	0.0005	0.085	0.0006	0.219	0.0051	0.234	0.0072
		12	0.089	0.0005	0.085	0.0005	0.219	0.0042	0.232	0.0055
30	9	10	0.086	0.0004	0.084	0.0005	0.223	0.0086	0.245	0.0174
		12	0.088	0.0004	0.085	0.0005	0.218	0.0060	0.234	0.0089
		14	0.087	0.0004	0.084	0.0005	0.222	0.0048	0.237	0.0067
40	5	6	0.087	0.0005	0.084	0.0005	0.214	0.0060	0.230	0.0093
		8	0.088	0.0005	0.084	0.0005	0.216	0.0035	0.228	0.0042
		10	0.089	0.0005	0.084	0.0005	0.220	0.0028	0.230	0.0033
40	7	8	0.086	0.0004	0.084	0.0004	0.215	0.0068	0.233	0.0114
		10	0.087	0.0004	0.084	0.0004	0.216	0.0039	0.230	0.0051
		12	0.088	0.0004	0.084	0.0004	0.220	0.0033	0.232	0.0041
40	9	10	0.085	0.0003	0.084	0.0003	0.217	0.0074	0.236	0.0134
		12	0.087	0.0003	0.084	0.0003	0.216	0.0046	0.231	0.0061
		14	0.087	0.0003	0.085	0.0003	0.219	0.0039	0.232	0.0048

6 Simulations and Data Analysis

6.1 Simulation Results

In this section we present some simulation results to see how the BE works for different sample sizes and for different values of τ_1 and τ_2. Along with the coverage percentage (CP)

Table 2: CP and AL of 90% CRIs and CI for λ_1 and λ_2 based on 5000 simulations with $\lambda_1 = 0.083$, $\lambda_2 = 0.222$, $a = 0.001$, $b = 0.001$, $c = 1$, and $d = 1$ for the Type-I censored case.

			λ_1						λ_2					
			Symm. CRI		HPD CRI		Boot. CI		Symm. CRI		HPD CRI		Boot. CI	
n	τ_1	τ_2	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL
10	5	6	92.62	0.140	89.88	0.132	88.14	0.122	98.10	0.550	96.38	0.472	98.54	0.694
		8	93.04	0.137	89.22	0.130	89.28	0.126	91.78	0.361	86.66	0.332	92.12	0.461
		10	94.58	0.138	90.46	0.131	89.60	0.128	90.56	0.332	87.14	0.310	91.52	0.438
10	7	8	89.92	0.124	87.34	0.117	84.22	0.111	97.80	0.598	94.38	0.503	98.44	0.848
		10	91.54	0.124	89.06	0.118	85.92	0.118	90.94	0.404	87.44	0.363	92.02	0.592
		12	92.08	0.124	90.62	0.118	86.32	0.119	90.86	0.368	87.56	0.338	91.64	0.532
10	9	10	91.10	0.115	87.08	0.109	82.04	0.103	98.30	0.679	97.84	0.560	97.52	1.114
		12	91.44	0.114	89.62	0.109	83.78	0.109	94.20	0.449	89.36	0.396	95.26	0.733
		14	90.92	0.115	90.62	0.110	84.62	0.112	92.62	0.407	86.78	0.366	92.76	0.669
20	5	6	90.22	0.100	89.14	0.097	84.28	0.098	95.32	0.356	88.90	0.321	92.42	0.403
		8	91.74	0.100	91.44	0.097	85.54	0.099	90.42	0.259	86.98	0.246	90.10	0.293
		10	91.62	0.100	91.22	0.097	84.82	0.098	89.58	0.228	86.06	0.220	89.60	0.259
20	7	8	90.24	0.089	88.98	0.087	89.74	0.089	95.92	0.390	89.98	0.346	96.34	0.454
		10	91.14	0.089	90.16	0.087	90.12	0.090	89.88	0.277	84.84	0.260	88.66	0.323
		12	92.20	0.090	90.72	0.088	89.96	0.090	90.16	0.251	86.98	0.239	89.96	0.294
20	9	10	90.24	0.082	88.74	0.080	88.08	0.081	96.88	0.425	92.24	0.370	98.22	0.514
		12	91.22	0.082	90.44	0.080	88.90	0.083	91.32	0.297	85.90	0.276	90.14	0.359
		14	92.04	0.083	91.50	0.081	89.84	0.084	89.42	0.272	86.00	0.256	89.06	0.333
30	5	6	90.42	0.083	89.08	0.081	85.92	0.082	91.88	0.300	85.56	0.276	88.94	0.334
		8	91.50	0.083	90.18	0.081	86.62	0.082	88.34	0.215	84.50	0.207	87.50	0.236
		10	91.18	0.084	90.24	0.082	86.86	0.083	90.18	0.193	87.08	0.187	90.40	0.210
30	7	8	90.12	0.073	88.78	0.072	87.60	0.074	92.68	0.318	86.44	0.288	90.88	0.362
		10	90.66	0.074	90.64	0.073	88.42	0.075	89.24	0.233	84.84	0.222	88.42	0.260
		12	90.92	0.074	90.76	0.073	88.04	0.074	88.98	0.206	85.38	0.199	88.68	0.229
30	9	10	89.96	0.068	89.74	0.067	88.96	0.068	94.90	0.341	88.80	0.304	93.30	0.398
		12	90.58	0.068	90.54	0.067	88.72	0.069	89.00	0.248	84.06	0.235	87.70	0.283
		14	91.32	0.068	90.98	0.067	88.64	0.069	89.50	0.224	86.18	0.215	88.74	0.255
40	5	6	91.60	0.072	90.64	0.071	88.78	0.072	90.46	0.265	84.30	0.248	88.06	0.292
		8	90.52	0.073	90.30	0.072	87.60	0.072	89.08	0.190	85.48	0.185	88.30	0.205
		10	90.56	0.073	90.78	0.072	88.06	0.072	89.06	0.168	86.98	0.165	88.96	0.180
40	7	8	89.98	0.064	89.26	0.063	88.00	0.065	91.16	0.282	84.40	0.260	87.76	0.313
		10	91.00	0.064	90.68	0.063	88.02	0.065	89.14	0.205	84.92	0.198	88.16	0.223
		12	90.68	0.065	90.54	0.064	88.00	0.065	88.70	0.182	85.88	0.178	88.52	0.198
40	9	10	90.00	0.059	89.16	0.058	88.12	0.059	92.44	0.300	85.06	0.272	90.12	0.339
		12	90.76	0.059	90.22	0.058	89.06	0.060	89.50	0.220	84.68	0.210	88.56	0.243
		14	90.74	0.060	90.52	0.059	88.60	0.060	88.90	0.196	85.16	0.189	88.76	0.216

and average length (AL) of symmetric CRI, HPD CRI, same of bootstrap CI is also presented for a comparison purpose. Here we choose $\lambda_1 = 1/12 \backsimeq 0.083$ and $\lambda_2 = 1/4.5 \backsimeq 0.222$. We also choose $a = 0.001$, $b = 0.001$, $c = 1$ and $d = 1$, *i.e.*, the non-informative prior and hence, the comparison with MLE is meaningful. All the results are based on 5000 simulations and

Table 3: CP of credible set for $(\alpha,\ \lambda_2)$ based on 5000 simulations with $\lambda_1 = 0.083$, $\lambda_2 = 0.222$, $a = 0.001$, $b = 0.001$, $c = 1$, and $d = 1$ for the Type-I censored case.

	$\tau_1 = 5$				$\tau_1 = 7$				$\tau_1 = 9$			
		CP				CP				CP		
n	τ_2	90%	95%	99%	τ_2	90%	95%	99%	τ_2	90%	95%	99%
10	6	87.00	90.84	98.10	8	87.20	91.92	97.52	10	86.12	94.92	98.96
	8	86.78	91.16	97.72	10	84.98	93.42	98.46	12	87.18	93.12	98.22
	10	86.12	92.50	98.74	12	85.96	93.64	98.78	14	87.88	93.70	98.54
20	6	85.52	93.66	98.74	8	86.98	93.58	98.80	10	86.62	93.58	98.62
	8	85.48	92.50	98.24	10	84.74	92.80	98.76	12	84.88	92.52	98.56
	10	84.66	91.76	98.46	12	85.02	92.38	98.70	14	84.84	92.34	98.48
30	6	84.26	91.98	98.46	8	85.66	93.44	98.96	10	85.42	94.08	99.08
	8	84.46	92.28	98.38	10	85.30	92.44	98.58	12	84.94	92.24	98.46
	10	85.64	92.72	98.50	12	87.00	93.20	98.60	14	85.78	92.44	98.86
40	6	83.92	92.20	98.40	8	83.76	92.58	98.58	10	85.38	93.36	98.68
	8	86.12	92.54	98.30	10	85.14	92.14	98.70	12	85.54	92.72	98.80
	10	86.82	93.06	98.44	12	86.82	93.24	98.74	14	86.46	93.28	98.70

$M = 8000$. We chose $n = 10$ (small sample size), 20 (moderate sample size), 30, and 40 (large sample size). For Type-II progressive censoring scheme, we choose $R_1 = n-m$, $R_i = 0$ for $i = 2,\ 3,\ ldots,\ m$. Note that Type-II censoring is a special case of Type-II progressive censoring scheme with $m = r$, $R_i = 0$, for $i = 1,\ 2,\ \dots,\ m-1$ and $R_m = n-m$. In all the calculation we have discarded those samples for whose the BEs or MLEs are greater than 100 times of the true value of the parametric function concerned. We have noticed that the most of the cases number of these type samples is zero. However the number of these type of samples are greater than zero in some cases corresponding to small values of n, or small values of expected failures in any one of the stress levels. The maximum percentage of these type of samples is 0.38%, which is noticed in case of HCS-I with $n = 10$, $\tau_1 = 9$, $\tau_2 = 10$, and $r = 7$. Average estimate (AE) and mean squared errors (MSE) of BE along with that of MLE for λ_1 and λ_2 are presented in the Tables 1, 4, 6, 8 and 10 for different values of n, τ_1, τ_2, and r. The CP and AL of 90% symmetric CRI, HPD CRI, and bootstrap CI for same n, τ_1, τ_2, r are reported in the Tables 2, 5, 7, 9, and 11. Interested readers are referred to supplementary file for more simulation results. We also report the CP of the HPD credible set for $(\alpha,\ \lambda_2)$ in the Table 3 using Algorithm 5.2 for the same parametric values when the data is Type-I censored.

The following points are quite clear from the simulation results. MSE of estimator of

Table 4: AE and MSE of MLE and BE of λ_1 and λ_2 based on 5000 simulations with $\lambda_1 = 0.083$, $\lambda_2 = 0.222$, $a = 0.001$, $b = 0.001$, $c = 1$, and $d = 1$ for the Type-II censored case.

			λ_1				λ_2			
			BE		MLE		BE		MLE	
n	τ_1	r	AE	MSE	AE	MSE	AE	MSE	AE	MSE
10	5	7	0.096	0.0015	0.087	0.0018	0.287	0.0743	0.366	0.4819
		8	0.098	0.0018	0.089	0.0021	0.267	0.0332	0.308	0.1650
		9	0.098	0.0018	0.089	0.0022	0.255	0.0212	0.281	0.0428
10	7	7	0.086	0.0008	0.079	0.0010	0.298	0.0925	0.430	0.7997
		8	0.092	0.0012	0.085	0.0014	0.282	0.0554	0.357	0.3116
		9	0.094	0.0014	0.087	0.0016	0.266	0.0307	0.309	0.1055
10	9	7	0.079	0.0005	0.073	0.0007	0.306	0.1298	0.496	1.1530
		8	0.087	0.0008	0.081	0.0009	0.296	0.0876	0.422	0.6378
		9	0.093	0.0012	0.087	0.0013	0.282	0.0728	0.355	0.3691
20	5	14	0.092	0.0009	0.086	0.0010	0.243	0.0119	0.267	0.0234
		16	0.092	0.0010	0.085	0.0011	0.234	0.0068	0.251	0.0103
		18	0.093	0.0010	0.086	0.0011	0.232	0.0058	0.246	0.0083
20	7	14	0.089	0.0007	0.084	0.0007	0.255	0.0255	0.301	0.1442
		16	0.091	0.0008	0.086	0.0009	0.244	0.0177	0.270	0.0407
		18	0.092	0.0008	0.087	0.0009	0.235	0.0076	0.253	0.0119
20	9	14	0.084	0.0004	0.081	0.0005	0.278	0.0635	0.378	0.5069
		16	0.089	0.0006	0.085	0.0006	0.253	0.0262	0.300	0.1858
		18	0.090	0.0006	0.086	0.0007	0.239	0.0100	0.263	0.0203
30	5	21	0.091	0.0007	0.085	0.0007	0.232	0.0056	0.248	0.0087
		24	0.091	0.0007	0.085	0.0007	0.227	0.0041	0.240	0.0055
		27	0.091	0.0007	0.085	0.0007	0.225	0.0032	0.236	0.0041
30	7	21	0.089	0.0005	0.085	0.0005	0.240	0.0117	0.265	0.0255
		24	0.089	0.0005	0.084	0.0005	0.230	0.0057	0.247	0.0085
		27	0.090	0.0005	0.085	0.0005	0.227	0.0043	0.240	0.0058
30	9	21	0.086	0.0003	0.083	0.0004	0.256	0.0341	0.316	0.2168
		24	0.088	0.0004	0.085	0.0005	0.240	0.0127	0.269	0.0583
		27	0.089	0.0005	0.085	0.0005	0.227	0.0053	0.243	0.0078
40	5	28	0.089	0.0005	0.085	0.0005	0.227	0.0042	0.240	0.0057
		32	0.089	0.0005	0.084	0.0005	0.225	0.0029	0.236	0.0037
		36	0.090	0.0005	0.085	0.0005	0.224	0.0024	0.233	0.0029
40	7	28	0.088	0.0004	0.085	0.0004	0.233	0.0106	0.255	0.0703
		32	0.088	0.0004	0.085	0.0004	0.226	0.0043	0.240	0.0059
		36	0.089	0.0004	0.085	0.0004	0.224	0.0028	0.234	0.0034
40	9	28	0.086	0.0003	0.083	0.0003	0.246	0.0233	0.287	0.1559
		32	0.088	0.0003	0.085	0.0004	0.231	0.0061	0.250	0.0095
		36	0.088	0.0003	0.084	0.0003	0.224	0.0037	0.237	0.0049

λ_1 decreases as τ_1 increases when other parameters are held constant. MSE of estimator of λ_2 decreases as τ_2 increases, whereas it increases as τ_1 increases. Also MSEs of all the estimators decrease as n increases. MSE of MLE is close to that of BE for λ_1, but MSE of MLE is higher than MSE of BE for λ_2. This difference decreases as expected number of failures at second stress level increases.

The performance of CI and CRIs of λ_1 are not satisfactory for all small sample sizes,

Table 5: CP and AL of 90% CRIs and CI for λ_1 and λ_2 based on 5000 simulations with $\lambda_1 = 0.083$, $\lambda_2 = 0.222$, $a = 0.001$, $b = 0.001$, $c = 1$, and $d = 1$ for the Type-II censored case.

			λ_1						λ_2					
			Symm. CRI		HPD CRI		Boot. CI		Symm. CRI		HPD CRI		Boot. CI	
n	τ_1	r	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL
10	5	7	97.28	0.140	92.72	0.133	90.26	0.110	93.58	0.515	90.78	0.449	94.34	1.069
		8	95.08	0.141	91.36	0.134	87.84	0.122	93.14	0.410	91.30	0.373	94.58	0.712
		9	95.24	0.139	91.36	0.132	88.54	0.128	91.72	0.346	90.28	0.323	93.68	0.512
10	7	7	96.78	0.120	92.64	0.114	88.26	0.088	93.56	0.617	90.12	0.513	93.68	1.497
		8	95.32	0.123	94.04	0.118	88.32	0.104	93.44	0.498	90.94	0.435	93.84	1.018
		9	93.54	0.124	93.04	0.119	86.40	0.116	93.00	0.406	90.66	0.369	93.70	0.699
10	9	7	94.82	0.105	91.88	0.100	83.42	0.072	93.60	0.709	88.54	0.569	92.36	1.916
		8	95.56	0.112	93.16	0.107	85.64	0.089	93.26	0.603	89.62	0.503	93.50	1.463
		9	93.32	0.116	93.54	0.112	85.42	0.105	93.42	0.492	90.96	0.430	93.96	1.009
20	5	14	92.32	0.102	91.90	0.099	84.60	0.097	91.86	0.297	90.14	0.280	94.22	0.413
		16	91.44	0.101	91.30	0.098	84.04	0.099	91.36	0.248	89.08	0.238	92.54	0.309
		18	92.00	0.102	91.62	0.099	85.26	0.100	90.16	0.224	88.38	0.217	91.58	0.263
20	7	14	93.86	0.090	91.94	0.087	90.18	0.080	92.58	0.377	89.70	0.339	93.64	0.640
		16	92.00	0.091	90.98	0.088	89.04	0.088	91.84	0.301	89.84	0.282	93.30	0.429
		18	91.50	0.091	90.72	0.089	89.46	0.091	90.98	0.252	89.22	0.241	92.00	0.317
20	9	14	95.76	0.081	93.56	0.079	91.02	0.065	92.34	0.509	89.22	0.431	93.22	1.047
		16	92.84	0.083	92.14	0.081	89.36	0.077	92.46	0.368	89.18	0.331	93.92	0.623
		18	91.78	0.084	92.10	0.082	89.18	0.083	91.44	0.286	89.34	0.269	93.30	0.394
30	5	21	91.54	0.085	90.24	0.083	86.58	0.083	91.36	0.231	88.96	0.222	92.64	0.283
		24	91.12	0.085	89.98	0.083	86.60	0.083	90.06	0.200	88.24	0.194	91.68	0.229
		27	91.44	0.085	90.46	0.083	86.92	0.083	89.96	0.180	88.22	0.176	91.12	0.199
30	7	21	91.90	0.075	92.04	0.073	87.38	0.071	91.62	0.288	89.08	0.269	93.68	0.406
		24	91.54	0.074	91.16	0.073	87.52	0.074	90.88	0.230	88.44	0.221	92.10	0.281
		27	91.08	0.075	91.24	0.074	88.82	0.075	90.06	0.200	88.14	0.195	91.04	0.230
30	9	21	94.50	0.068	93.94	0.067	91.14	0.058	92.28	0.395	88.84	0.347	93.74	0.688
		24	90.96	0.068	90.96	0.067	86.98	0.067	90.78	0.285	88.28	0.266	92.44	0.404
		27	90.76	0.069	90.78	0.068	88.46	0.069	89.92	0.222	87.14	0.214	91.88	0.269
40	5	28	90.24	0.074	90.58	0.073	87.82	0.073	89.86	0.197	88.04	0.192	91.04	0.227
		32	90.52	0.074	91.02	0.073	88.28	0.073	90.44	0.173	88.50	0.169	91.08	0.190
		36	89.96	0.074	90.34	0.073	87.96	0.073	89.98	0.156	88.28	0.153	90.74	0.168
40	7	28	90.34	0.065	90.88	0.064	86.52	0.063	91.08	0.245	88.16	0.232	92.96	0.321
		32	90.76	0.065	90.48	0.064	87.96	0.065	90.12	0.198	87.40	0.192	91.66	0.228
		36	90.26	0.066	89.92	0.065	87.74	0.065	90.60	0.172	88.46	0.169	91.74	0.190
40	9	28	93.18	0.059	92.72	0.058	89.96	0.053	91.66	0.327	88.30	0.297	93.46	0.501
		32	90.82	0.060	90.76	0.059	87.90	0.059	91.32	0.234	88.92	0.224	93.06	0.293
		36	90.56	0.060	90.36	0.059	88.56	0.060	89.60	0.192	87.34	0.186	91.86	0.219

but they are quite satisfactory for moderate and large sample sizes. We note that average length of HPD CRI of λ_1 decreases as τ_1 or n increases, keeping the other fixed. HPD CRI of λ_1 performs well compared to the symmetric CRI and bootstrap CI with respect to CP of the respective intervals, though AL of HPD CRI is larger than that of the bootstrap CI for

Table 6: AE and MSE of MLE and BE of λ_1 and λ_2 based on 5000 simulations with $\lambda_1 = 0.083$, $\lambda_2 = 0.222$, $a = 0.001$, $b = 0.001$, $c = 1$, and $d = 1$ for the Type-I hybrid censored case.

				λ_1				λ_2			
				BE		MLE		BE		MLE	
n	τ_1	τ_1	r	AE	MSE	AE	MSE	AE	MSE	AE	MSE
10	5	6	7	0.0865	0.0012	0.0707	0.0013	0.3599	0.0771	0.4998	0.2237
		8	7	0.0882	0.0011	0.0781	0.0013	0.2773	0.0488	0.3397	0.1426
		10	7	0.0897	0.0010	0.0806	0.0013	0.2771	0.1103	0.3369	0.3403
10	7	8	7	0.0772	0.0007	0.0641	0.0010	0.4055	0.1571	0.6072	0.5292
		10	7	0.0779	0.0006	0.0687	0.0008	0.2787	0.0469	0.3632	0.1477
		12	7	0.0778	0.0005	0.0696	0.0008	0.2700	0.0639	0.3491	0.2164
10	9	10	7	0.0688	0.0006	0.0575	0.0010	0.4264	0.1873	0.6744	0.7619
		12	7	0.0684	0.0005	0.0601	0.0009	0.2880	0.1070	0.4057	0.4027
		14	7	0.0687	0.0005	0.0611	0.0008	0.2720	0.0847	0.3755	0.3370
20	5	6	14	0.0871	0.0009	0.0816	0.0010	0.2487	0.0133	0.2880	0.0260
		8	14	0.0899	0.0009	0.0848	0.0010	0.2359	0.0161	0.2573	0.0371
		10	14	0.0903	0.0009	0.0841	0.0010	0.2398	0.0105	0.2613	0.0185
20	7	8	14	0.0840	0.0006	0.0790	0.0006	0.2732	0.0292	0.3380	0.0781
		10	14	0.0861	0.0005	0.0820	0.0006	0.2460	0.0288	0.2823	0.0755
		12	14	0.0866	0.0005	0.0821	0.0006	0.2486	0.0190	0.2848	0.0472
20	9	10	14	0.0785	0.0004	0.0738	0.0004	0.3024	0.0927	0.4014	0.2554
		12	14	0.0801	0.0003	0.0763	0.0004	0.2610	0.0679	0.3233	0.2707
		12	14	0.0803	0.0003	0.0764	0.0004	0.2584	0.0355	0.3179	0.1059
30	5	6	21	0.0864	0.0006	0.0832	0.0007	0.2212	0.0075	0.2415	0.0125
		8	21	0.0883	0.0006	0.0837	0.0007	0.2221	0.0066	0.2360	0.0098
		10	21	0.0899	0.0006	0.0844	0.0007	0.2301	0.0059	0.2453	0.0087
30	7	10	21	0.0874	0.0005	0.0840	0.0005	0.2331	0.0100	0.2557	0.0185
		12	21	0.0881	0.0004	0.0843	0.0005	0.2377	0.0107	0.2611	0.0195
		14	21	0.0879	0.0005	0.0841	0.0005	0.2364	0.0125	0.2602	0.0243
30	9	10	21	0.0819	0.0003	0.0792	0.0003	0.2529	0.0264	0.3046	0.0708
		12	21	0.0836	0.0003	0.0807	0.0003	0.2453	0.0197	0.2846	0.0515
		14	21	0.0835	0.0003	0.0804	0.0003	0.2521	0.0297	0.2951	0.0797
40	5	6	28	0.0864	0.0005	0.0838	0.0005	0.2147	0.0062	0.2303	0.0097
		8	28	0.0879	0.0005	0.0836	0.0005	0.2204	0.0045	0.2321	0.0058
		10	28	0.0892	0.0005	0.0844	0.0005	0.2266	0.0042	0.2388	0.0057
40	7	8	28	0.0856	0.0004	0.0835	0.0004	0.2202	0.0078	0.2398	0.0135
		10	28	0.0874	0.0004	0.0841	0.0004	0.2280	0.0064	0.2458	0.0104
		12	28	0.0872	0.0004	0.0837	0.0004	0.2322	0.0074	0.2504	0.0121
40	9	10	28	0.0834	0.0003	0.0815	0.0003	0.2353	0.0140	0.2675	0.0293
		12	28	0.0850	0.0002	0.0825	0.0003	0.2368	0.0133	0.2658	0.0283
		14	28	0.0855	0.0002	0.0829	0.0003	0.2432	0.0145	0.2753	0.0341

Type-I, Type-II, and hybrid Type-I censoring schemes, but smaller than that of symmetric CRI. For moderate and large sample sizes, AL of symmetric CRI, HPD CRI and bootstrap CI are very close to each other for all the censoring schemes.

The performance of the HPD CRI is not so satisfactory for λ_2 with respect to CP. However, CP of symmetric CRI and bootstrap CI is close to nominal level when expected number of failures at first stress level is large. For small sample size performance of CRIs as

Table 7: CP and AL of 90% CRIs and CI for λ_1 and λ_2 based on 5000 simulations with $\lambda_1 = 0.083$, $\lambda_2 = 0.222$, $a = 0.001$, $b = 0.001$, $c = 1$, and $d = 1$ for the Type-I hybrid censored case.

				λ_1						λ_2					
				Symm. CRI		HPD CRI		Boot. CI		Symm. CRI		HPD CRI		Boot. CI	
n	τ_1	τ_2	r	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL
10	5	6	7	98.98	0.137	88.50	0.123	87.10	0.106	97.40	0.761	99.70	0.578	97.24	1.228
		8	7	97.50	0.133	89.92	0.114	89.80	0.108	94.04	0.488	89.80	0.369	91.12	1.013
		10	7	98.34	0.134	91.40	0.109	90.00	0.109	92.08	0.475	87.70	0.343	91.46	1.033
10	7	8	7	94.86	0.114	85.02	0.103	84.64	0.086	97.90	0.901	99.52	0.645	94.88	2.010
		10	7	94.60	0.112	89.02	0.097	87.64	0.088	95.78	0.530	91.28	0.380	91.42	1.520
		12	7	95.70	0.111	90.16	0.093	87.20	0.088	91.26	0.505	88.54	0.351	91.70	1.597
10	9	10	7	90.30	0.098	85.32	0.089	78.12	0.072	98.82	1.000	99.86	0.694	93.40	2.907
		12	7	91.30	0.095	85.54	0.084	82.46	0.073	96.38	0.599	90.86	0.407	90.16	2.115
		14	7	92.44	0.095	87.40	0.082	81.80	0.072	92.90	0.549	88.10	0.377	91.18	2.000
20	5	6	14	92.08	0.100	89.68	0.092	84.14	0.095	97.80	0.399	95.90	0.337	92.14	0.473
		8	14	91.88	0.100	91.44	0.087	83.50	0.095	89.82	0.299	85.98	0.239	89.26	0.426
		10	14	92.58	0.100	91.98	0.084	83.22	0.096	90.96	0.289	88.76	0.230	91.24	0.408
20	7	8	14	93.24	0.087	90.20	0.081	90.10	0.080	98.12	0.477	97.58	0.372	95.36	0.788
		10	14	96.00	0.087	91.66	0.077	90.78	0.080	91.30	0.359	85.38	0.264	90.90	0.690
		12	14	95.28	0.088	91.94	0.074	91.24	0.080	91.44	0.353	88.50	0.258	92.76	0.668
20	9	10	14	93.26	0.077	89.18	0.072	89.74	0.065	98.42	0.578	99.00	0.431	95.12	1.128
		12	14	95.04	0.078	91.80	0.068	90.04	0.065	91.70	0.439	86.62	0.290	91.66	1.039
		12	14	95.48	0.078	92.16	0.066	91.12	0.065	92.28	0.427	88.22	0.291	93.06	1.008
30	5	6	21	91.08	0.082	88.96	0.079	86.50	0.082	95.20	0.305	89.14	0.278	89.10	0.348
		8	21	92.22	0.083	90.28	0.080	86.04	0.082	88.84	0.230	83.82	0.218	88.10	0.293
		10	21	91.56	0.084	90.62	0.081	86.98	0.083	90.20	0.229	87.66	0.218	91.06	0.285
30	7	10	21	93.02	0.073	92.06	0.071	87.26	0.070	89.60	0.281	84.94	0.260	91.62	0.443
		12	21	93.06	0.074	92.52	0.072	87.00	0.070	91.30	0.281	88.44	0.261	89.82	0.422
		14	21	92.24	0.074	92.00	0.072	87.12	0.071	91.48	0.280	88.60	0.260	93.62	0.442
30	9	10	21	93.68	0.065	91.00	0.064	91.16	0.058	97.78	0.417	94.78	0.363	91.92	0.732
		12	21	95.44	0.066	93.46	0.064	91.26	0.058	90.94	0.351	86.26	0.314	92.20	0.675
		14	21	95.58	0.066	93.82	0.064	91.26	0.058	91.44	0.361	87.32	0.323	94.30	0.718
40	5	6	28	90.94	0.072	89.52	0.070	87.94	0.072	91.14	0.267	84.28	0.246	87.82	0.295
		8	28	91.42	0.073	90.52	0.070	87.66	0.072	88.50	0.200	84.20	0.191	88.72	0.239
		10	28	91.90	0.074	91.46	0.071	87.58	0.073	90.40	0.197	87.38	0.189	91.06	0.231
40	7	8	28	90.78	0.064	89.60	0.062	86.82	0.063	93.50	0.292	85.86	0.265	88.22	0.351
		10	28	91.10	0.064	90.16	0.063	87.42	0.063	89.76	0.240	85.70	0.226	89.82	0.319
		12	28	92.10	0.064	91.64	0.063	86.94	0.063	90.42	0.240	87.40	0.226	92.28	0.318
40	9	10	28	93.80	0.058	92.20	0.056	89.94	0.053	94.88	0.343	88.96	0.305	90.30	0.505
		12	28	94.38	0.058	92.90	0.057	89.72	0.053	90.06	0.301	85.82	0.274	91.50	0.553
		14	28	94.68	0.059	93.90	0.057	90.26	0.053	91.98	0.311	87.94	0.283	94.24	0.502

well as bootstrap CI for λ_2 not at all satisfactory, specially when expected number of failures at first stress level is small. ALs of CRIs and bootstrap CI of λ_2 decrease as expected number of failures at second stress level or n increases, keeping the other parameter constant. Also note that ALs of bootstrap CI and HPD CRI of the same parameter increase as τ_1 increases.

Table 8: AE and MSE of MLE and BE of λ_1 and λ_2 based on 5000 simulations with $\lambda_1 = 0.083$, $\lambda_2 = 0.222$, $a = 0.001$, $b = 0.001$, $c = 1$, and $d = 1$ for the Type-II hybrid censored case.

				λ_1				λ_2			
				BE		MLE		BE		MLE	
n	τ_1	τ_1	r	AE	MSE	AE	MSE	AE	MSE	AE	MSE
10	5	6	7	0.091	0.0012	0.081	0.0014	0.274	0.0726	0.327	0.2148
		8	7	0.095	0.0015	0.085	0.0018	0.260	0.0180	0.295	0.0373
		10	7	0.096	0.0016	0.088	0.0021	0.250	0.0151	0.274	0.0304
10	7	8	7	0.081	0.0007	0.073	0.0009	0.274	0.0434	0.349	0.1240
		10	7	0.089	0.0011	0.081	0.0013	0.266	0.0219	0.314	0.0514
		12	7	0.092	0.0013	0.085	0.0015	0.258	0.0399	0.294	0.1358
10	9	10	7	0.074	0.0006	0.066	0.0008	0.300	0.1008	0.415	0.3451
		12	7	0.084	0.0009	0.077	0.0010	0.279	0.0372	0.346	0.1020
		14	7	0.089	0.0011	0.082	0.0012	0.262	0.0303	0.308	0.0704
20	5	6	14	0.091	0.0009	0.085	0.0011	0.241	0.0099	0.262	0.0169
		8	14	0.091	0.0009	0.085	0.0011	0.231	0.0061	0.248	0.0093
		10	14	0.092	0.0009	0.086	0.0011	0.226	0.0050	0.239	0.0069
20	7	8	14	0.088	0.0006	0.083	0.0007	0.252	0.0204	0.288	0.0555
		10	14	0.091	0.0007	0.086	0.0008	0.236	0.0079	0.257	0.0132
		12	14	0.090	0.0007	0.085	0.0009	0.225	0.0056	0.240	0.0084
20	9	10	14	0.084	0.0005	0.080	0.0005	0.263	0.0271	0.316	0.0779
		12	14	0.089	0.0007	0.086	0.0007	0.226	0.0073	0.243	0.0118
		12	14	0.088	0.0006	0.084	0.0007	0.231	0.0078	0.254	0.0141
30	5	6	21	0.090	0.0006	0.084	0.0007	0.230	0.0054	0.246	0.0079
		8	21	0.090	0.0007	0.084	0.0007	0.228	0.0040	0.241	0.0055
		10	21	0.090	0.0007	0.085	0.0007	0.223	0.0033	0.234	0.0042
30	7	10	21	0.089	0.0005	0.085	0.0006	0.225	0.0047	0.240	0.0068
		12	21	0.089	0.0005	0.085	0.0006	0.221	0.0042	0.233	0.0055
		14	21	0.089	0.0005	0.084	0.0006	0.222	0.0037	0.234	0.0047
30	9	10	21	0.086	0.0004	0.083	0.0004	0.243	0.0121	0.275	0.0260
		12	21	0.088	0.0004	0.085	0.0005	0.223	0.0054	0.240	0.0082
		14	21	0.088	0.0005	0.085	0.0005	0.222	0.0051	0.236	0.0071
40	5	6	28	0.089	0.0005	0.084	0.0005	0.228	0.0039	0.240	0.0052
		8	28	0.088	0.0005	0.083	0.0005	0.225	0.0032	0.237	0.0041
		10	28	0.089	0.0005	0.084	0.0005	0.220	0.0025	0.229	0.0029
40	7	8	28	0.088	0.0004	0.084	0.0004	0.232	0.0065	0.249	0.0099
		10	28	0.087	0.0004	0.084	0.0004	0.222	0.0036	0.236	0.0048
		12	28	0.088	0.0004	0.084	0.0004	0.219	0.0031	0.229	0.0037
40	9	10	28	0.087	0.0003	0.084	0.0003	0.235	0.0083	0.260	0.0161
		12	28	0.087	0.0003	0.084	0.0003	0.219	0.0041	0.233	0.0057
		14	28	0.088	0.0003	0.084	0.0003	0.219	0.0036	0.231	0.0045

6.2 Data Analysis

Example 1

Here we consider the data (see Table 12) presented by Xiong [18] to illustrate the methods of estimation discussed previously. This is actually a Type-II censored data from a simple step stress life experiment, where $n = 20$ units are placed on the test, the data is right

Table 9: CP and AL of 90% CRIs and CI for λ_1 and λ_2 based on 5000 simulations with $\lambda_1 = 0.083$, $\lambda_2 = 0.222$, $a = 0.001$, $b = 0.001$, $c = 1$, and $d = 1$ for the Type-II hybrid censored case.

				λ_1						λ_2					
				Symm. CRI		HPD CRI		Boot. CI		Symm. CRI		HPD CRI		Boot. CI	
n	τ_1	τ_2	r	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL
10	5	6	7	98.12	0.135	92.72	0.098	90.10	0.122	93.42	0.457	90.52	0.346	94.28	0.590
		8	7	96.86	0.138	92.12	0.104	88.52	0.127	92.20	0.397	90.08	0.318	96.30	0.451
		10	7	96.28	0.139	92.36	0.108	89.40	0.128	92.24	0.352	90.28	0.290	94.96	0.416
10	7	8	7	95.62	0.115	91.06	0.091	87.54	0.107	92.60	0.504	88.70	0.352	94.26	0.749
		10	7	94.34	0.122	91.94	0.089	86.86	0.117	92.54	0.437	89.10	0.336	96.66	0.541
		12	7	94.00	0.124	92.28	0.092	86.62	0.119	93.06	0.394	91.20	0.304	93.62	0.533
10	9	10	7	93.22	0.101	89.16	0.079	84.00	0.097	92.20	0.607	88.46	0.409	93.90	0.954
		12	7	93.30	0.110	90.82	0.079	84.82	0.108	92.78	0.493	90.30	0.366	96.66	0.738
		14	7	93.82	0.113	92.46	0.080	84.76	0.111	93.92	0.426	92.24	0.325	93.16	0.668
20	5	6	14	92.44	0.101	92.18	0.085	85.64	0.099	91.64	0.288	90.32	0.225	92.62	0.348
		8	14	92.16	0.101	93.32	0.085	85.36	0.100	91.32	0.261	88.68	0.218	94.20	0.277
		10	14	92.22	0.101	92.80	0.085	84.38	0.098	91.38	0.234	89.54	0.203	92.02	0.253
20	7	8	14	94.38	0.089	92.38	0.075	90.64	0.089	92.34	0.356	89.32	0.263	94.44	0.417
		10	14	92.56	0.091	92.38	0.075	89.84	0.090	92.10	0.293	89.06	0.237	94.70	0.309
		12	14	92.10	0.090	92.46	0.079	90.52	0.090	92.14	0.251	89.72	0.215	91.02	0.286
20	9	10	14	93.84	0.080	92.04	0.066	89.56	0.080	93.20	0.420	89.24	0.300	95.48	0.477
		12	14	91.38	0.083	90.52	0.080	89.78	0.084	92.16	0.271	87.50	0.253	93.06	0.348
		12	14	92.38	0.083	91.28	0.080	89.02	0.084	93.10	0.310	89.28	0.284	90.40	0.325
30	5	6	21	92.68	0.084	91.24	0.081	87.22	0.084	91.48	0.228	89.22	0.217	92.08	0.265
		8	21	91.60	0.084	90.34	0.082	87.22	0.083	91.08	0.216	88.92	0.207	92.32	0.223
		10	21	91.74	0.085	90.44	0.082	87.40	0.083	91.90	0.194	89.40	0.186	90.00	0.203
30	7	10	21	91.42	0.075	91.04	0.072	88.54	0.074	91.86	0.236	88.04	0.223	92.88	0.318
		12	21	91.58	0.074	91.16	0.072	88.12	0.075	90.46	0.208	86.36	0.198	92.54	0.246
		14	21	91.00	0.075	90.14	0.072	87.72	0.075	90.36	0.195	86.70	0.187	88.60	0.226
30	9	10	21	93.66	0.068	92.20	0.066	90.30	0.068	92.16	0.332	88.34	0.301	94.76	0.367
		12	21	92.08	0.068	91.34	0.067	88.08	0.069	92.54	0.254	88.34	0.237	89.96	0.274
		14	21	91.04	0.068	90.58	0.066	89.26	0.069	90.12	0.224	85.48	0.212	88.36	0.251
40	5	6	28	90.88	0.074	90.00	0.072	88.64	0.073	90.96	0.198	88.72	0.189	90.64	0.221
		8	28	90.84	0.074	90.34	0.071	87.30	0.073	90.28	0.189	87.70	0.182	92.44	0.191
		10	28	90.88	0.074	90.22	0.072	87.80	0.073	91.28	0.169	88.38	0.163	89.72	0.174
40	7	8	28	91.18	0.065	90.42	0.063	87.56	0.065	90.46	0.238	87.30	0.225	92.64	0.268
		10	28	91.36	0.065	90.54	0.063	87.76	0.065	90.36	0.207	87.28	0.198	91.70	0.213
		12	28	91.10	0.065	90.34	0.063	88.24	0.065	89.38	0.182	85.76	0.174	89.46	0.194
40	9	10	28	92.02	0.059	91.28	0.058	89.54	0.060	92.52	0.287	88.82	0.264	94.12	0.316
		12	28	91.22	0.059	90.92	0.058	88.38	0.060	92.16	0.223	86.84	0.210	89.58	0.235
		14	28	90.96	0.060	90.44	0.058	88.02	0.060	90.18	0.197	85.78	0.188	89.04	0.215

censored at 16th failure time and stress changing time is $\tau_1 = 5$. The average lifetimes are 3.52 and 7.70 at first and second stress level, whareas standard deviations are 1.05 and 1.72.Balakrishnan et al. [8] used this data for illustrative example for step-stress model under Type-I censoring scheme choosing $\tau_2 = 7$, 8, 9, and 12. They reported the MLE of $1/\lambda_1$ and $1/\lambda_2$ and associated CIs for above mentioned τ_1 and τ_2's under the assumption that the

Table 10: AE and MSE of MLE and BE of λ_1 and λ_2 based on 5000 simulations with $\lambda_1 = 0.083$, $\lambda_2 = 0.222$, $a = 0.001$, $b = 0.001$, $c = 1$, and $d = 1$ for the Type-II progressive censored case.

			λ_1				λ_2			
			BE		MLE		BE		MLE	
n	τ_1	m	AE	MSE	AE	MSE	AE	MSE	AE	MSE
10	5	7	0.101	0.0028	0.091	0.0033	0.270	0.0333	0.307	0.1299
		8	0.100	0.0023	0.090	0.0027	0.261	0.0266	0.290	0.0924
		9	0.100	0.0020	0.090	0.0024	0.251	0.0143	0.272	0.0253
10	7	7	0.098	0.0022	0.090	0.0025	0.283	0.0627	0.340	0.2851
		8	0.097	0.0018	0.089	0.0021	0.275	0.0584	0.320	0.1813
		9	0.097	0.0017	0.089	0.0020	0.261	0.0258	0.294	0.0652
10	9	7	0.097	0.0019	0.090	0.0021	0.296	0.0897	0.390	0.5946
		8	0.095	0.0016	0.089	0.0018	0.284	0.0637	0.357	0.3815
		9	0.096	0.0015	0.089	0.0017	0.272	0.0382	0.322	0.1336
20	5	14	0.095	0.0013	0.087	0.0015	0.235	0.0071	0.251	0.0105
		16	0.095	0.0011	0.087	0.0013	0.234	0.0059	0.248	0.0084
		18	0.093	0.0010	0.085	0.0012	0.231	0.0048	0.244	0.0066
20	7	14	0.093	0.0011	0.087	0.0013	0.239	0.0097	0.258	0.0158
		16	0.093	0.0009	0.087	0.0011	0.236	0.0074	0.253	0.0113
		18	0.092	0.0008	0.086	0.0009	0.231	0.0060	0.247	0.0089
20	9	14	0.093	0.0010	0.088	0.0011	0.243	0.0179	0.269	0.0423
		16	0.092	0.0008	0.087	0.0009	0.238	0.0093	0.260	0.0156
		18	0.091	0.0007	0.086	0.0008	0.236	0.0080	0.255	0.0131
30	5	21	0.093	0.0009	0.086	0.0010	0.229	0.0039	0.241	0.0053
		24	0.092	0.0008	0.085	0.0009	0.227	0.0035	0.237	0.0045
		27	0.092	0.0007	0.085	0.0008	0.227	0.0032	0.237	0.0041
30	7	21	0.091	0.0007	0.086	0.0008	0.230	0.0053	0.245	0.0075
		24	0.091	0.0007	0.086	0.0007	0.229	0.0042	0.242	0.0057
		27	0.091	0.0006	0.085	0.0006	0.227	0.0040	0.239	0.0052
30	9	21	0.091	0.0007	0.086	0.0007	0.232	0.0065	0.250	0.0106
		24	0.091	0.0006	0.086	0.0006	0.229	0.0052	0.245	0.0075
		27	0.089	0.0005	0.085	0.0005	0.227	0.0045	0.242	0.0062
40	5	28	0.092	0.0007	0.086	0.0007	0.224	0.0028	0.233	0.0035
		32	0.091	0.0007	0.085	0.0007	0.224	0.0025	0.233	0.0030
		36	0.090	0.0006	0.084	0.0006	0.224	0.0022	0.232	0.0026
40	7	28	0.090	0.0006	0.085	0.0006	0.225	0.0034	0.236	0.0044
		32	0.090	0.0005	0.085	0.0005	0.226	0.0032	0.237	0.0041
		36	0.090	0.0005	0.086	0.0005	0.226	0.0029	0.236	0.0036
40	9	28	0.089	0.0005	0.085	0.0005	0.226	0.0044	0.240	0.0060
		32	0.089	0.0004	0.085	0.0004	0.226	0.0039	0.239	0.0051
		36	0.089	0.0004	0.085	0.0004	0.227	0.0036	0.238	0.0046

data are coming form exponential CEM. Choosing $a = 0.001$, $b = 0.001$, $c = 1$, $d = 1$ and $M = 8000$, the BE and Bayesian CRIs for $1/\lambda_1$ and $1/\lambda_2$ are reported in Tables 13 and 14, respectively. We also obtain HPD credible set for (λ_1, λ_2) and is given by

$$\mathcal{C}_\gamma = \left\{ (\alpha\lambda_2,\, \lambda_2) \in \mathbb{R}^2 : \frac{c_0}{\Gamma(n_2+4)} \alpha^4 \, \lambda_2^{n_2+3} \, e^{\,94.07\alpha + D_2} \geq c_\gamma \right\},$$

Table 11: CP and AL of 90% CRIs and CI for λ_1 and λ_2 based on 5000 simulations with $\lambda_1 = 0.083$, $\lambda_2 = 0.222$, $a = 0.001$, $b = 0.001$, $c = 1$, and $d = 1$ for the Type-II progressively censored data.

			λ_1						λ_2					
			Symm. CRI		HPD CRI		Boot. CI		Symm. CRI		HPD CRI		Boot. CI	
n	τ_1	m	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL
10	5	7	88.94	0.156	82.06	0.147	79.86	0.150	92.52	0.414	91.38	0.379	93.78	0.687
		8	92.54	0.149	86.38	0.141	85.06	0.143	92.38	0.368	91.26	0.343	93.68	0.560
		9	95.52	0.144	89.14	0.137	88.82	0.136	92.94	0.327	91.80	0.308	93.44	0.460
10	7	7	89.12	0.144	91.16	0.137	89.60	0.138	92.58	0.480	90.92	0.427	93.88	0.932
		8	92.02	0.137	93.20	0.130	86.68	0.133	92.58	0.430	90.58	0.391	93.70	0.750
		9	91.80	0.131	93.04	0.125	83.72	0.128	91.90	0.378	90.20	0.348	93.50	0.606
10	9	7	92.92	0.137	93.30	0.130	85.16	0.128	93.06	0.559	90.36	0.480	93.72	1.321
		8	93.78	0.128	92.52	0.122	87.66	0.123	92.62	0.496	90.24	0.436	93.56	1.034
		9	93.04	0.122	92.06	0.117	87.32	0.120	92.74	0.436	90.86	0.391	94.46	0.821
20	5	14	92.22	0.116	91.18	0.112	87.02	0.116	91.40	0.250	89.76	0.240	92.02	0.304
		16	92.64	0.111	92.62	0.108	84.64	0.111	91.16	0.232	89.36	0.224	92.14	0.274
		18	92.18	0.105	90.72	0.102	85.12	0.104	91.04	0.216	89.86	0.210	91.92	0.250
20	7	14	91.68	0.105	91.10	0.102	89.08	0.107	91.00	0.276	88.88	0.263	92.68	0.359
		16	91.82	0.100	91.92	0.097	89.68	0.101	91.24	0.254	89.46	0.244	92.10	0.316
		18	91.68	0.095	92.36	0.092	87.38	0.095	90.80	0.236	89.00	0.227	91.98	0.284
20	9	14	91.72	0.099	91.32	0.096	87.48	0.101	91.78	0.310	89.44	0.290	94.00	0.442
		16	92.60	0.093	92.12	0.090	90.92	0.095	91.50	0.280	89.08	0.265	92.68	0.371
		18	91.54	0.088	91.40	0.086	88.00	0.089	91.04	0.262	88.90	0.249	92.28	0.333
30	5	21	91.30	0.099	91.76	0.096	88.38	0.099	91.12	0.201	89.76	0.195	91.58	0.227
		24	90.68	0.093	90.62	0.091	85.92	0.093	90.58	0.186	88.68	0.182	91.54	0.207
		27	91.72	0.089	90.84	0.087	91.50	0.088	90.24	0.176	88.56	0.173	90.62	0.193
30	7	21	90.84	0.088	91.02	0.086	87.60	0.089	90.32	0.219	88.60	0.212	91.72	0.257
		24	89.84	0.083	90.06	0.081	88.40	0.084	91.10	0.204	89.94	0.198	91.44	0.234
		27	91.58	0.079	91.40	0.077	88.92	0.079	90.34	0.192	88.96	0.187	91.04	0.216
30	9	21	90.78	0.081	90.46	0.080	87.90	0.083	90.72	0.242	88.80	0.232	92.58	0.296
		24	90.64	0.077	90.88	0.075	88.12	0.078	90.84	0.223	88.26	0.215	91.64	0.264
		27	90.58	0.072	89.96	0.071	88.10	0.073	90.40	0.208	88.24	0.202	91.66	0.241
40	5	28	90.70	0.087	90.90	0.085	87.88	0.087	90.96	0.171	88.96	0.168	91.82	0.187
		32	89.90	0.082	89.74	0.080	88.70	0.081	90.32	0.160	89.20	0.157	90.48	0.173
		36	90.88	0.078	90.74	0.076	88.18	0.077	89.98	0.151	88.68	0.149	91.14	0.162
40	7	28	91.22	0.077	90.34	0.076	89.24	0.077	90.44	0.187	88.80	0.182	91.74	0.209
		32	90.44	0.073	90.44	0.071	88.56	0.073	89.98	0.176	87.96	0.172	90.58	0.194
		36	89.28	0.069	89.70	0.068	87.88	0.069	89.84	0.167	88.54	0.163	90.12	0.181
40	9	28	91.78	0.071	91.16	0.070	90.42	0.072	90.44	0.204	87.92	0.198	91.52	0.235
		32	90.04	0.067	89.96	0.066	88.34	0.067	90.04	0.192	87.80	0.187	91.14	0.217
		36	89.30	0.063	89.28	0.062	88.88	0.063	89.78	0.182	87.74	0.177	90.72	0.202

Table 12: Data of Example 1.

Stress Level	Failure Times					
$\lambda_1 = e^{-2.5}$	2.01	3.60	4.12	4.34		
$\lambda_2 = e^{-1.5}$	5.04	5.94	6.68	7.09	7.17	7.49
	7.60	8.23	8.24	8.25	8.69	12.05

Table 13: Estimates of $1/\lambda_1$ and associated CRIs and bootstrap CI for the data in Table 12.

				Symm. CRI		HPD CRI		Boot. CI	
Level	τ_2	BE	MLE	LL	UL	LL	UL	LL	UL
90%	7	27.404	23.517	12.216	55.026	8.652	45.198	10.899	49.082
	8	24.117	23.517	10.864	48.544	8.147	40.144	10.993	49.252
	9	23.680	23.517	10.442	49.268	7.871	39.945	10.743	49.208
	12	23.840	23.517	10.654	48.353	8.471	40.391	10.743	49.208
95%	7	-	-	10.862	66.069	8.575	55.837	10.896	97.806
	8	-	-	9.727	59.088	8.097	50.062	10.901	98.291
	9	-	-	9.359	59.925	6.764	49.833	11.019	98.609
	12	-	-	9.504	59.737	7.335	49.265	11.019	98.608
99%	7	-	-	8.741	98.123	6.552	84.455	9.498	99.708
	8	-	-	8.001	90.597	5.838	77.397	9.319	99.969
	9	-	-	7.624	88.249	5.943	73.701	9.206	99.969
	12	-	-	7.805	89.867	5.825	76.321	9.206	99.969

Table 14: Estimates of $1/\lambda_2$ and associated CRIs and bootstrap CI for the data in Table 12.

				Symm. CRI		HPD CRI		Boot. CI	
Level	τ_2	BE	MLE	LL	UL	LL	UL	LL	UL
90%	7	13.136	9.553	5.248	26.514	3.791	22.319	3.346	19.100
	8	7.391	5.573	3.682	13.553	2.947	11.811	2.394	10.115
	9	5.028	4.129	2.890	8.380	2.453	7.475	2.074	6.487
	12	6.623	5.493	3.825	10.849	3.515	10.056	3.015	8.446
95%	7	-	-	4.519	31.011	3.396	26.922	3.102	24.735
	8	-	-	3.315	15.442	2.820	13.958	2.109	12.270
	9	-	-	2.646	9.319	2.207	8.461	1.947	7.537
	12	-	-	3.549	12.331	3.159	11.174	2.627	9.284
99%	7	-	-	3.502	45.710	2.367	37.127	2.982	35.586
	8	-	-	2.720	19.837	2.318	18.339	1.895	17.534
	9	-	-	2.234	12.056	2.036	11.025	1.613	9.844
	12	-	-	2.979	14.810	2.592	14.081	2.313	12.041

Table 15: Credible set of (λ_1, λ_2) for data in Table 12.

τ_2	n_2	D_2	c_0	$c_{0.90}$	$c_{0.95}$	$c_{0.99}$
7	3	28.66	3.97630×10^{10}	0.000342718	0.000184043	0.000042354
8	7	39.01	4.28721×10^{18}	0.000302052	0.000170764	0.000037351
9	11	45.42	3.45450×10^{26}	0.000275559	0.000126912	0.000032489
12	11	60.42	6.09132×10^{27}	0.000309364	0.000164498	0.000047736

where c_0, n_2, and D_2 depend on τ_2 and are presented in Table 15. Figure 2 shows the plot of the HPD credible set of (λ_1, λ_2) for different values of τ_2.

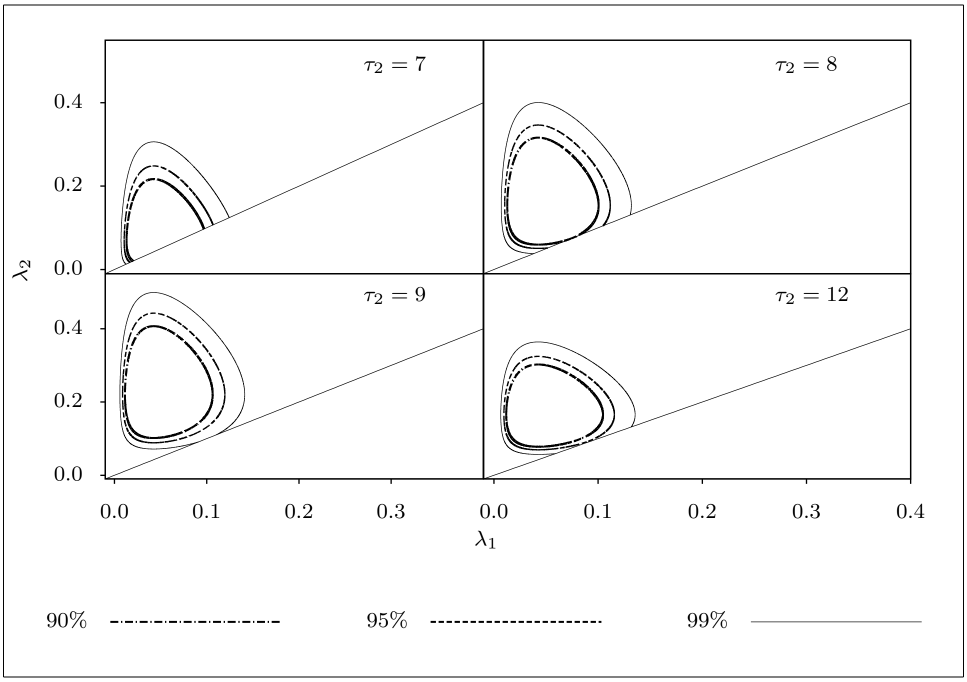


Figure 2: Credible set of (λ_1, λ_2) for the data in Table 12.

Table 16: Data of Example 2.

Stress Level	Failure Times								
$\lambda_1 = e^{-3.5}$	1.46	2.22	3.92	4.24	5.47	5.60	6.12	6.57	
$\lambda_2 = e^{-2.0}$	8.19	8.30	8.74	8.98	9.43	9.87	11.14	11.76	11.85
	12.14	13.05	13.49	14.04	14.19	14.24	14.33	15.28	16.58
	16.85	16.92	17.80	20.45	20.98	21.09	22.01	26.34	28.66

Example 2

Next, we consider the data (see Table 16) presented by Balakrishnan et al. [8]. Choice made by Balakrishnan et al. [8] were $n = 35$, $\lambda_1 = e^{-3.5}$, $\lambda_2 = e^{-2.0}$ and $\tau_1 = 8$. This is a complete data set. Average lifetimes are 4.45 and 15.06 at the first and second stress levels, respectively. Standard deviations are 1.85 and 5.41, respectively. To make it a Type-I censored data one may take any τ_2 greater than 8. Balakrishnan et al. [8] took different choices for τ_2, *viz.*, 16, 20 and 24. Along these choices of τ_2, we take $\tau_2 = 12$ also. Assuming that the data are coming form the exponential CEM under Type-I censoring, they presented MLE and associated CIs of $1/\lambda_1$ and $1/\lambda_2$. Here we present the BE of $1/\lambda_1$ and $1/\lambda_2$ and associated CRIs for the above mentioned values of τ_2 under the same assumption. The results are presented in Tables 17 and 18 with $a = 0.001$, $b = 0.001$, $c = 1$, $d = 1$, and $M = 8000$.

Like the previous example, we also obtain HPD credible set for $(\lambda_1,\, \lambda_2)$ and is given by

$$\mathcal{C}_\gamma = \left\{ (\alpha\lambda_2,\, \lambda_2) \in \mathbb{R}^2 : \frac{c_0}{\Gamma(n_2+8)} \alpha^8\, \lambda_2^{n_2+7}\, e^{\,251.60\alpha + D_2} \geq c_\gamma \right\},$$

where c_0, n_2, and D_2 depend on τ_2 and are presented in Table 19. Figure 3 shows the plot of the HPD credible set of $(\lambda_1,\, \lambda_2)$ for different values of τ_2.

Table 17: Estimates of $1/\lambda_1$ and associated CRIs and bootstrap CI for data in Table 16.

				Symm. CRI		HPD CRI		Boot. CI	
Level	τ_2	BE	MLE	LL	UL	LL	UL	LL	UL
90%	12	31.945	31.450	18.042	54.526	15.602	48.254	16.670	53.347
	16	31.537	31.450	17.508	53.279	15.368	47.685	16.487	53.425
	20	31.490	31.450	17.462	53.157	14.922	47.228	16.051	53.356
	24	31.353	31.450	17.481	53.269	14.695	47.212	16.587	53.420
95%	12	-	-	16.348	61.683	14.239	55.525	16.473	67.078
	16	-	-	16.057	60.810	13.679	54.299	16.487	67.146
	20	-	-	15.966	61.189	13.901	53.968	16.533	67.398
	24	-	-	15.925	61.083	13.678	54.050	16.484	67.356
99%	12	-	-	13.867	81.539	12.244	74.194	13.499	91.582
	16	-	-	13.578	80.729	10.957	71.613	12.871	90.735
	20	-	-	13.712	78.961	11.804	71.838	12.804	91.596
	24	-	-	13.666	79.610	11.468	70.773	12.804	91.885

Table 18: Estimates of $1/\lambda_2$ and associated CRIs and bootstrap CI for data in Table 16.

				Symm. CRI		HPD CRI		Boot. CI	
Level	τ_2	BE	MLE	LL	UL	LL	UL	LL	UL
90%	12	12.169	9.807	6.584	20.589	5.675	18.644	4.943	16.583
	16	9.490	8.413	6.148	14.217	5.629	13.258	5.029	11.900
	20	9.010	8.151	6.096	12.910	5.504	12.025	5.343	11.328
	24	7.991	7.348	5.621	11.122	5.260	10.584	4.906	9.945
95%	12	-	-	6.056	23.062	5.249	20.946	4.586	19.153
	16	-	-	5.757	15.394	5.419	14.736	4.845	13.374
	20	-	-	5.722	13.989	5.479	13.360	4.836	12.066
	24	-	-	5.269	12.148	4.953	11.439	4.695	10.789
99%	12	-	-	5.143	28.593	3.924	26.374	4.224	26.035
	16	-	-	4.980	18.444	4.497	17.143	4.196	15.698
	20	-	-	5.050	16.179	4.655	15.324	4.325	14.410
	24	-	-	4.746	13.888	4.635	13.436	3.903	12.069

Table 19: Credible set of $(\lambda_1,\, \lambda_2)$ for data in Table 16.

τ_2	n_2	D_2	c_0	$c_{0.90}$	$c_{0.95}$	$c_{0.99}$
12	9	88.26	1.98906×10^{38}	0.000634529	0.000363521	0.000084468
16	17	143.02	1.83135×10^{59}	0.000683449	0.000368794	0.000030347
20	21	171.17	1.46811×10^{70}	0.000751840	0.000368533	0.000087536
24	25	183.70	2.77355×10^{80}	0.000804461	0.000399712	0.000078281

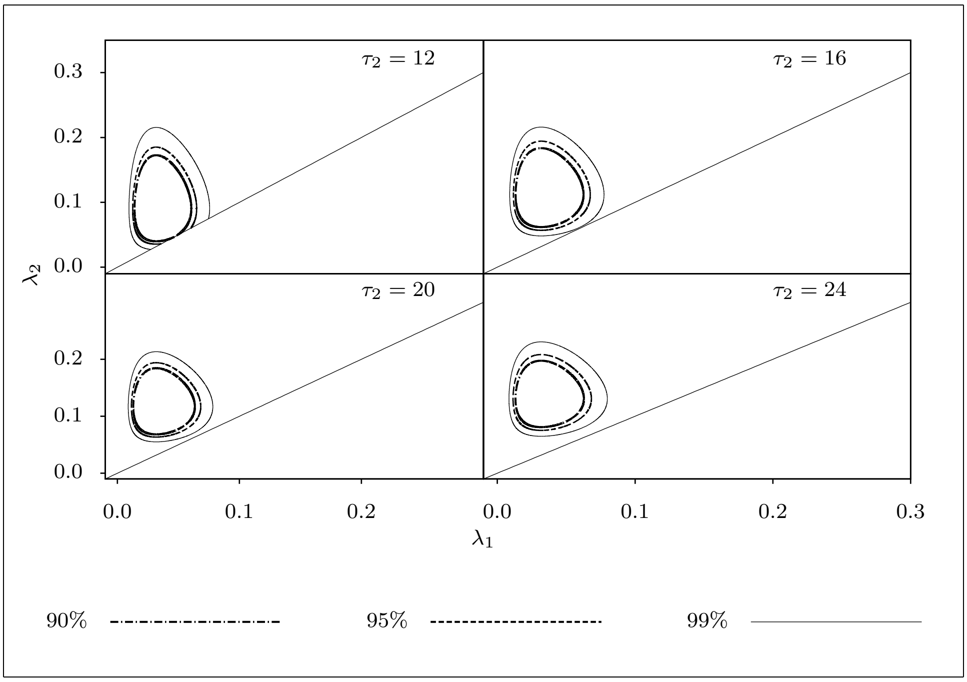


Figure 3: Credible set of (λ_1, λ_2) for the data in Table 16.

7 Conclusion

We have considered the Bayesian estimation of the unknown parameters in a simple SSLT under the restriction $\lambda_1 < \lambda_2$ and under different censoring schemes. The analysis is performed under exponentially distributed lifetimes and under CEM assumption. We have taken mainly the squared error loss function, though other loss functions can also be handled in a very similar way. We have seen that the BE of some parametric function under the square error loss function does not exist in close form in most of the cases. Algorithms based on importance sampling are proposed to compute BE and to construct CRIs and credible set. We have done a simulation study to judge the performance of the procedures described. We also considered two data sets to illustrate the estimation procedures. We have noticed that the performance of BE and CRIs for λ_1 is quite satisfactory at least for moderate and large sample sizes. It is also noticed that the performance of BE and CRI for λ_1 is better than that of MLE and bootstrap CI for small sample size and for the small values of τ_1. For moderate and large sample sizes the performance of BE and CRI is quite close to that of MLE and bootstrap CI. We have also noticed that the performance of BE, CRI, MLE and bootstrap CI of λ_2 are not satisfactory for small values of n and small values of expected number of failures at the second stress level. However, BE and CRI work quite well for moderate or large sample sizes and for large values of expected number of failures at the second stress level and the performance of BE and CRI is close to that of MLE and bootstrap CI of λ_2 in these cases. It is also noticed that HPD CRI works well for λ_1, where symmetric CRI works well for λ_2. Therefore we recommend to use HPD CRI for λ_1 and symmetric CRI for λ_2.

Note that one may generate α form other distributions having support on (0,1) instead of uniform distribution as mentioned in Algorithm 5.1. A right truncated gamma distribution and the distribution obtained by spline fitting to the posterior distribution of α have been tried. However, no significant improvement has been noticed. However, the results are quite prior dependent. The choice of priors is an important issue, which has not been pursued here and more research is needed in this direction.

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