Analysis of Simple Step-stress Model in Presence of Competing Risks

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Abstract

In this article, we consider a simple step-stress life test in the presence of exponentially distributed competing risks. It is assumed that the stress is changed when a pre-specified number of failures takes place. The data is assumed to be Type-II censored. We obtain the maximum likelihood estimators of the model parameters and the exact conditional distributions of the maximum likelihood estimators. Based on the conditional distribution, approximate confidence intervals of unknown parameters have been constructed. Percentile bootstrap confidence intervals of model parameters are also provided. Optimal test plan is addressed. We perform an extensive simulation study to observe the behavior of the proposed method. The performances are quite satisfactory. Finally we analyze two data sets for illustrative purposes.

Keywords: Step-stress life test, Censoring, Competing risks, Maximum likelihood Estimation, Bootstrap confidence interval.

1 Introduction

Now a days, most of the products are highly reliable due to severe competitiveness in the market. The experimenter experimenting with such a product faces the problem of very few failures or no failure at all, in an affordable time. The accelerated life tests (ALT) are proposed to overcome this problem. In an accelerated life testing experiment, extreme stress levels are imposed on the product under consideration to ensure rapid failures. Interested

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readers are referred to Nelson [41] and Bagdanavicius and Nikulin [3] for an exposure to different ALT models.

The step-stress life test (SSLT), where the experimenter is allowed to change the stress levels during the experiment, is a special case of ALT. In a conventional step stress experiment, the stress levels are changed at pre-fixed time points. The data collected from such a SSLT, may then be extrapolated to estimate the underlying distribution of failure times under normal stress level. This process requires a model relating the levels of stress and the failure time distributions. Several such models are proposed in the literature. Cumulative exposure model (CEM) is most studied in the literature which was originally proposed by Seydyakin [42]. SSLT with exponentially distributed lifetimes under CEM is extensively studied by several authors. Interested readers are referred to an excellent review article by Balakrishnan [6] in this respect. Balakrishnan et al. [13] and Balakrishnan et al. [12] considered the SSLT under the exponential CEM in the presence of Type-I and Type-II censoring schemes, respectively. SSLT under the exponential CEM was addressed by Balakrishnan and Xie [11] and Balakrishnan and Xie [10] under hybrid Type-I and hybrid Type-II censoring schemes, respectively. The optimality issues of a SSLT under exponential CEM was addressed by several authors including Miller and Nelson [40], Bai et al. [5], Gouno et al. [26], Han et al. [29], Leu and Shen [37], Xie et al. [45], Fan et al. [24], Balakrishnan and Han [7], Wu et al. [44], Yuan and Liu [48] in the presence of different censoring schemes. Optimal step-stress test under Type-I censoring for multivariate exponential distribution was discussed by Guan and Tang [27]. Alkhalfan [2] considered the SSLT in the presence of different censoring schemes when the distribution of lifetime is assumed to be gamma distribution. Several inferential issues along with optimality of SSLT for the Weibull distributed lifetime can be found in Bai and Kim [4], Kateri and Balakrishnan [32], and Liu [39]. Chung and Bai [18], Alhadeed and Yang [1], Balakrishnan et al. [14], and Lin and Chou [38] studied the SSLT under the assumption of log-normal CEM in the presence of several censoring schemes. Ebraheim and Al-Masri [22] and Ismail [31] addressed inferential issues for SSLT under the assumption of log-logistic and generalized exponential lifetimes, respectively. Fard and Li [25] and Hunt and Xu [30] studied optimality issues in designing SSLT for reliability prediction.

It is very common that the product under consideration is exposed to more than one causes in a life testing experiment. In such a situation, one needs to assess the effect of each cause in the presence of other causes. In the statistical literature, it is known as the competing risks problem. Extensive work has been done on the analysis of competing risks data both in the parametric and non-parametric set-up. See for example, Cox [19], David and Moeschberger [21], Crowder [20], and the references therein for different issues related to competing risks problems. The ALT in the presence of competing risks can be found in Klein and Basu [33, 34]. Balakrishnan and Han [8] considered the simple SSLT model in the presence of competing risks, when the competing causes of failures have exponential distribution.

One major drawback of the classical SSLT is that the model parameters are estimable if there is at least one failure at each stress level. To overcome that problem, Xiong and Milliken [46], Xiong et al. [47], Wang and Yu [43], and Kundu and Balakrishnan [35] considered the following model. Suppose n items are put on the test at the initial stress level s_1. The stress level is changed to the next stress level s_2 as soon as the r_1-th failure occurs. Similarly, the stress level is changed to s_3 form s_2 as soon as the $(r_1 + r_2)$-th failure occurs. In general, the stress level is changed to s_{i+1} from s_i at the time of occurrence of $(r_1 + r_2 + \ldots + r_i)$-th failure for $i = 1, 2, \ldots, k-1$. Here $r_1, r_2, \ldots, r_{k-1}$ are pre-fixed positive integers such that $r_1 + \ldots + r_{k-1} \leq n$.

The main aim of this article is to consider a simple SSLT, under the above experimental setup, in the presence of competing risks. The following assumptions are made: (i) There are two causes of failures and the model satisfies the latent failure time model assumption of Cox [19]. (ii) The latent failure times are exponentially distributed with scale parameter θ_{ij} at the stress level s_i, $i = 1, 2$, in the presence of jth, $j = 1, 2$, cause only. (iii) The latent failure times are independently distributed for both the causes. (iv) The data are Type-II censored. (v) The CEM assumptions hold for each of the latent failure distributions under the whole step-stress pattern. Though an extensive discussion on exponential CEM can be found in Balakrishnan [6], the exponential CEM with random stress changing time was not addressed in this article at all. Balakrishnan and Han [8] studied the exponential CEM in the presence of competing risks and Type-II censoring, where it is assumed that the stress levels are changed at pre-fixed time points. The main advantage of the proposed model is that it is analytically more tractable than the models proposed in Balakrishnan [6] or Balakrishnan and Han [8].

We obtain the conditional maximum likelihood estimators (MLEs) of all the unknown parameters and their exact distributions. We discuss an approximate confidence interval (ACI) and bootstrap confidence interval (BCI) of the model parameters based on the distributions of MLEs. Optimal tests are obtained based on D-optimality and A-optimality criteria. We have performed extensive simulation experiments to observe the performance of the MLEs and the associated CIs. Analysis of two data set are provided for illustrative purpose. Rest of the article is organized as follows. In the Section 2, we discuss the model under consideration and obtain MLEs of the unknown parameters. Exact conditional distribution of MLEs of models parameters are considered in Section 3. The ACIs and BCIs of the unknown parameters are discussed in the Section 4. The Fisher information matrix and the optimal test plans based on the Fisher information matrix are studied in Section 5. Simulation and data analysis are provided in Section 6. Finally the article is concluded in Section 7. All the proofs are provided in the Appendix.

2 Model Description and MLE

Consider a simple step-stress life test with stress levels $s_1 < s_2$. Suppose that a sample of size n is put on the life testing experiment at the initial stress level s_1. The failure times are recorded and denoted by $t_{1:n} < t_{1:n} < \ldots < t_{n:n}$. Let r_1 and r_2 be two pre-fixed integers such that $0 < r_1 < n$, $0 < r_2 < n$, and $0 < r_1 + r_2 = r \leq n$. At the time of the r_1-th failure, $t_{r_1:n}$, stress level is changed form s_1 to next stress level s_2. Test is terminated as soon as the r-th failure occurs. It is assumed that there are only two causes of failure of an experimental item, and when an item fails, the failure time and the associated cause of failure are observed. Let the data be denoted by $(t_{i:n},\, \delta_{i:n})$, $i = 1,\, 2,\, \ldots,\, r$, where $t_{1:n} < \ldots < t_{r_1:n} < \ldots < t_{r:n}$, and

$$\delta_{i:n} = \begin{cases} 0 & \text{if } i\text{th failure occurs due to the first cause} \\ 1 & \text{if } i\text{th failure occurs due to the second cause.} \end{cases}$$

Let the corresponding random variables be denoted by $(T_{i:n},\, \Delta_{i:n})$, $i = 1,\, 2,\, \ldots,\, r$. We also assume that Cox's latent failure time model assumptions hold true here, and the latent failure times are independently exponentially distributed with mean θ_{ij} at stress level s_i, $i = 1,\, 2$, in the presence of j-th, $j = 1,\, 2$, cause only. Based on the CEM assumptions and following the same procedure as in Balakrishnan and Han [8], we can obtain the likelihood function as follows.

Let us denote $\boldsymbol{T}_{i,j} = (T_{i:n},\, \ldots,\, T_{j:n})$, $\boldsymbol{t}_{i,j} = (t_{i:n:},\, \ldots,\, t_{j:n})$, $\boldsymbol{\Delta}_{i,j} = (\Delta_{i:n},\, \ldots,\, \Delta_{j:n})$, $\boldsymbol{\delta}_{i,j} = (\delta_{i:n},\, \ldots,\, \delta_{j:n})$, and $c_{i,j,n} = \prod_{l=i}^{j} (n-l+1)$ for $1 \le i \le j \le r$. Denoting $\lambda_{ij} = \dfrac{1}{\theta_{ij}}$, $i = 1,\, 2,\; j = 1,\, 2$ and $D_1 = \displaystyle\sum_{i=1}^{r_1} t_{i:n} + (n - r_1)\, t_{r_1:n}$, the joint PDF of $\boldsymbol{T}_{1,r_1}$ and $\boldsymbol{\Delta}_{1,r_1}$ is given by

$$f_{\boldsymbol{T}_{1,r_1},\, \boldsymbol{\Delta}_{1,r_1}}(\boldsymbol{t}_{1,r_1},\, \boldsymbol{\delta}_{1,r_1}) = c_{1,r_1,n}\, \lambda_{11}^{\sum_{i=1}^{r_1} \delta_{i:n}}\, \lambda_{12}^{r_1 - \sum_{i=1}^{r_1} \delta_{i:n}}\, e^{-(\lambda_{11}+\lambda_{12}) D_1},$$

if $0 < t_{1:n} < \ldots < t_{r_1:n} < \infty$ and $\delta_{i:n} \in \{0,\, 1\}$ for all $i = 1,\, 2,\, \ldots,\, r_1$. Now using the lack of memory property of exponential distribution and denoting $D_2 = \displaystyle\sum_{i=r_1+1}^{r} (t_{i:n} - t_{r_1:n}) + (n - r)(t_{r:n} - t_{r_1:n})$, the conditional joint PDF of $\boldsymbol{T}_{r_1+1,r}$ and $\boldsymbol{\Delta}_{r_1+1,r}$ conditioning on $\boldsymbol{T}_{1,r_1}$ and $\boldsymbol{\Delta}_{1,r_1}$ is given by

$$f_{\boldsymbol{T}_{r_1+1,r},\, \boldsymbol{\Delta}_{r_1+1,r}}(\boldsymbol{t}_{r_1+1,r},\, \boldsymbol{\delta}_{r_1+1,r}) = c_{r_1+1,r,n}\, \lambda_{21}^{\sum_{i=r_1+1}^{r} \delta_{i:n}}\, \lambda_{22}^{r_2 - \sum_{i=r_1+1}^{r} \delta_{i:n}}\, e^{-(\lambda_{21}+\lambda_{22}) D_2},$$

if $t_{r_1:n} < t_{r_1+1:n} < t_{r_1+2:n} < \ldots < t_{r:n} < \infty$ and $\delta_{i:n} \in \{0,\, 1\}$ for all $i = r_1+1,\, r_1+2,\, \ldots,\, r$.

Hence the joint density function of $\boldsymbol{T}_{1,r}$ and $\boldsymbol{\Delta}_{1,r}$ is given by

$$f_{\boldsymbol{T}_{1,r},\,\boldsymbol{\Delta}_{1,r}}(\boldsymbol{t}_{1,r},\,\boldsymbol{\delta}_{1,r})$$
$$= c_{1,r,n}\,\lambda_{11}^{\sum_{i=1}^{r_1}\delta_{i:n}}\,\lambda_{12}^{r_1-\sum_{i=1}^{r_1}\delta_{i:n}}\,\lambda_{21}^{\sum_{i=r_1+1}^{r}\delta_{i:n}}\,\lambda_{22}^{r_2-\sum_{i=r_1+1}^{r}\delta_{i:n}}\,e^{-(\lambda_{11}+\lambda_{12})D_1-(\lambda_{21}+\lambda_{22})D_2}, \tag{1}$$

if $0 < t_{1:n} < \ldots < t_{r:n} < \infty$ and $\delta_{i:n} \in \{0,\,1\}$ for all $i = 1,\,2,\,\ldots,\,r$. Let $\boldsymbol{n} = (n_1,\,n_2)$, where n_1 and n_2 are the number of failures due to the first cause at the stress levels s_1 and s_2, respectively, and $\boldsymbol{N} = (N_1,\,N_2)$ is the corresponding random vector. The likelihood function of θ_{11}, θ_{12}, θ_{21}, and θ_{22} is given by

$$L(\theta_{11},\,\theta_{12},\,\theta_{21},\,\theta_{22}) \propto \theta_{11}^{-n_1}\,\theta_{12}^{-(r_1-n_1)}\,\theta_{21}^{-n_2}\,\theta_{22}^{-(r_2-n_2)}\,e^{-\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\right)D_1-\left(\frac{1}{\theta_{21}}+\frac{1}{\theta_{22}}\right)D_2}.$$

From the above likelihood function it is clear that the MLEs of all the known parameters exist if $\boldsymbol{n} \in \mathcal{N} = \{\boldsymbol{n} :\ 0 < n_1 < r_1,\ 0 < n_2 < r_2\}$. Whenever the MLEs exist, they are unique and are given by

$$\widehat{\theta}_{11} = \frac{D_1}{n_1}, \qquad \widehat{\theta}_{12} = \frac{D_1}{r_1-n_1}, \qquad \widehat{\theta}_{21} = \frac{D_2}{n_2}, \qquad \widehat{\theta}_{22} = \frac{D_2}{r_2-n_2}. \tag{2}$$

Clearly these are the conditional MLEs of the unknown parameters conditioned on the event that $\boldsymbol{N} \in \mathcal{N}$.

Remark 1. Though we have considered a simple SSLT with two competing risks in this article, it is fairly easy to extend the proposed model for multiple stress levels and for multiple competing risks.

3 Conditional Distribution of MLEs

In this section we will provide the conditional probability density functions (CPDF) of the MLEs conditioning on the event that $\boldsymbol{N} \in \mathcal{N}$. The conditional distribution can be used for construction of confidence intervals of the unknown parameters. The derivation of the

CPDFs requires the inversion of the conditional moment generating functions (CMGF) as first suggested by Bartholmew [15]. Moreover, in constructing the confidence intervals we need to prove certain monotonicity property of the distribution functions with respect to the parameter values at a fixed point. We provide an explicit proof of that using the Lemma 1 in Balakrishnan and Iliopoulos [9]. The statements of all the theorems are provided in the main text, and their proofs are given in the Appendix. Now we will define few notations which will be used in the theorems. The PDF and CDF of a gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$ are denoted by $f_{\mathrm{G}}(\cdot,\, \alpha,\, \beta)$ and $F_{\mathrm{G}}(\cdot,\, \alpha,\, \beta)$, and they are given by

$$f_G\,(x;\, \alpha,\, \beta) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-\beta\, x} & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases} \qquad F_G\,(x;\, \alpha,\, \beta) \;\; = \int_{-\infty}^{x} f_{\mathrm{G}}\,(t;\, \alpha,\, \beta)\, dt, \quad (3)$$

respectively. Also define $p_1 = \dfrac{\theta_{12}}{\theta_{11} + \theta_{12}}$, $p_2 = \dfrac{\theta_{22}}{\theta_{21} + \theta_{22}}$, $n_{i1k} = k$, $n_{i2k} = r_i - k$, $k = 1,\, 2,\, \ldots,\, r_i - 1$, and $i = 1,\, 2$.

Theorem 1. D_1 has a gamma distribution with PDF $f_G\left(\cdot;\, r_1,\, \dfrac{1}{\theta_{11}} + \dfrac{1}{\theta_{12}}\right)$.

Theorem 2. D_2 has a gamma distribution with PDF $f_G\left(\cdot;\, r_2,\, \dfrac{1}{\theta_{21}} + \dfrac{1}{\theta_{22}}\right)$.

Theorem 3. N_1 has a Binomial distribution with parameters r_1 and p_1, while N_2 has a Binomial distribution with parameters r_2 and p_2. Also N_1 and N_2 are independently distributed, *i.e.*,

$$P(\boldsymbol{N} = \boldsymbol{n}) = \binom{r_1}{n_1} p_1^{n_1}\, (1 - p_1)^{r_1 - n_1} \times \binom{r_2}{n_2} p_2^{n_2}\, (1 - p_2)^{r_2 - n_2}\,,$$
$$\text{if } n_1 = 0,\, 1,\, \ldots,\, r_1;\; n_2 = 0,\, 1,\, \ldots,\, r_2 - r_1.$$

Theorem 4. For $x \in \mathbb{R}$, the CPDF of $\widehat{\theta}_{ij}$, $i = 1,\, 2$, $j = 1,\, 2$, conditioning on the event $\boldsymbol{N} \in \mathcal{N}$ is given by

$$f_{\widehat{\theta}_{ij}}(x) = \frac{1}{1 - p_i^{r_i} - (1 - p_i)^{r_i}} \sum_{k=1}^{r_i - 1} \binom{r_i}{k} p_i^k\, (1 - p_i)^{r_i - k}\, f_{\mathrm{G}}\left(x;\, r_i,\, n_{ijk}\left(\frac{1}{\theta_{i1}} + \frac{1}{\theta_{i2}}\right)\right).$$

Corollary 1. For $x \in \mathbb{R}$, the conditional cumulative distribution function (CCDF) of $\widehat{\theta}_{ij}$, $i = 1,\, 2,\, j = 1,\, 2$, conditioning on the event $\boldsymbol{N} \in \mathcal{N}$ is given by

$$F_{\widehat{\theta}_{ij}}(x) = \frac{1}{1 - p_i^{r_i} - (1-p_i)^{r_i}} \sum_{k=1}^{r_i - 1} \binom{r_i}{k} p_i^k \,(1-p_i)^{r_i - k}\, F_{\mathrm{G}}\left(x;\, r_i,\, n_{ijk}\left(\frac{1}{\theta_{i1}} + \frac{1}{\theta_{i2}}\right)\right).$$

Corollary 2. The mean and the variance of $\widehat{\theta}_{ij}$, $i = 1,\, 2,\, j = 1,\, 2$, are given below.

$$\mathrm{E}\left(\widehat{\theta}_{ij}\right) = r_i \left(\frac{1}{\theta_{i1}} + \frac{1}{\theta_{i2}}\right)^{-1} \mathrm{E}\left(\frac{1}{X_{ij}} \,\middle|\, X_{ij} \neq 0,\, r_i\right)$$

and

$$\mathrm{Var}\left(\widehat{\theta}_{ij}\right) = r_i \left(\frac{1}{\theta_{i1}} + \frac{1}{\theta_{i2}}\right)^{-2} \mathrm{E}\left(\frac{1}{X_{ij}^2} \,\middle|\, X_{ij} \neq 0,\, r_i\right),$$

where $X_{i1} \sim \mathrm{Bin}\,(r_i,\, p_i)$ and $X_{i2} \sim \mathrm{Bin}\,(r_i,\, 1 - p_i)$.

Remark 2. Note that the model under consideration is quite general in the sense that it includes its marginal models as special cases. For example, if our aim is to estimate θ_{11} in case $\theta_{12} \to \infty$, the MLE of θ_{11} exists if $n_1 > 0$. Under the condition $n_1 > 0$, the PDF of the MLE of θ_{11} is given by

$$\tilde{f}_{\widehat{\theta}_{11}}(x) = \frac{1}{1 - (1-p_1)^{r_1}} \sum_{k=1}^{r_1} \binom{r_1}{k} p_1^k \,(1-p_1)^{r_1 - k}\, f_{\mathrm{G}}\left(x;\, r_1,\, k\left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right)\right),$$

which approaches to $f_{\mathrm{G}}(r_1, r_1)$ as $\theta_{12} \to \infty$ and this model becomes a simple step-stress model without competing risk, which was studied by Kundu and Balakrishnan [35]. The same holds in case any one the following holds; $\theta_{11} \to \infty$, $\theta_{21} \to \infty$ or $\theta_{22} \to \infty$.

4 Different Types of Confidence Intervals

In this section, we present different methods for construction of the confidence intervals (CIs) of the unknown parameters θ_{ij}, $i = 1,\, 2$ and $j = 1,\, 2$. From the Theorem 4, we can construct ACIs for θ_{ij}. However, as PDFs of θ_{ij}'s are quite complicated, we also present the BCIs for these unknown parameters.

4.1 Approximate Confidence Interval

For fixed x, let $F_{\widehat{\theta}_{ij}}(x, \theta_{ij})$ denote the CCDF of $\widehat{\theta}_{ij}$, $i = 1, 2$, $j = 1, 2$, as a function of θ_{ij} for fixed $\theta_{i'j'}$, $i' \neq i$ or $j' \neq j$. Next we state a theorem which will be used to construct ACI. The proof of the theorem is given in the Appendix.

Theorem 5. $F_{\widehat{\theta}_{ij}}(x, \theta_{ij})$ is a strictly decreasing function of θ_{ij} for all $x > 0$.

A two-sided $100(1-\alpha)\%$ ACI of θ_{ij} can be found based on the Corollary 1 and Theorem 5 and is given by $(\theta_{ij\mathrm{L}}, \theta_{ij\mathrm{U}})$, where $\theta_{ij\mathrm{L}}$ and $\theta_{ij\mathrm{U}}$ are the roots of the equations

$$F_{\widehat{\theta}_{ij}}(\widehat{\theta}_{ij\mathrm{obs}}, \theta_{ij\mathrm{L}}) = 1 - \frac{\alpha}{2} \qquad \text{and} \qquad F_{\widehat{\theta}_{ij}}(\widehat{\theta}_{ij\mathrm{obs}}, \theta_{ij\mathrm{U}}) = \frac{\alpha}{2}, \tag{4}$$

provided the equations are feasible. Note that $F_{\widehat{\theta}_{ij}}(\cdot, \cdot)$ involves other unknown parameters and we replace those parameters with their MLEs. However, (4) are non-linear equations, which can be solved using numerical procedures, *e.g.*, bisection method or Newton-Raphson method. This method of construction of confidence interval has been used by several authors including Chen and Bhattacharya [16], Gupta and Kundu [28], Kundu and Basu [36], Childs et al. [17] and Balakrishnan et al. [13]. Note that a one-sided $100(1-\alpha)\%$ ACI of θ_{ij} of the form $(0, \theta_{ij\mathrm{U}})$ can be found by solving the non-linear equation

$$F_{\widehat{\theta}_{ij}}(\widehat{\theta}_{ij\mathrm{obs}}, \theta_{ij\mathrm{U}}) = \alpha.$$

It may be noted that the one sided confidence interval of θ_{ij} can be constructed by solving one non-linear equation only.

4.2 Bootstrap Confidence Interval

Construction of ACIs for θ_{ij} as discussed in the previous subsection is computationally quite involved, specially when r_1 or r_2 is large. Hence in this subsection we consider the percentile BCI, see Efron and Tibshirani [23] for more details. We use the following algorithm to generate bootstrap sample and construct the BCI.

Algorithm 1

Step 1. Given n, r_1, r_2, and the original sample, obtain the MLEs of the unknown parameters $\widehat{\theta}_{11}$, $\widehat{\theta}_{12}$, $\widehat{\theta}_{21}$, and $\widehat{\theta}_{22}$ using (2).

Step 2. Generate d^*_{ij}, $i = 1, 2$, $j = 1, 2, \ldots, B$, from (3) with shape parameter $\alpha = r_i$ and scale parameter $\beta = \dfrac{1}{\widehat{\theta}_{i1}} + \dfrac{1}{\widehat{\theta}_{i2}}$.

Step 3. Generate n^*_{ij}, $i = 1, 2$, $j = 1, 2, \ldots, B$, form the following truncated Binomial distribution. For $x = 1, 2, \ldots, r_i - 1$

$$\frac{1}{1 - \widehat{p}_i^{r_i} - (1 - \widehat{p}_i)^{r_i}} \binom{r_i}{x} \widehat{p}_i^x \left(1 - \widehat{p}\right)^{r_i - x},$$

where $\widehat{p}_i = \dfrac{\widehat{\theta}_{i2}}{\widehat{\theta}_{i1} + \widehat{\theta}_{i2}}$.

Step 4. Set $\widehat{\theta}^*_{i1j} = \dfrac{d^*_{ij}}{n^*_{ij}}$ and $\widehat{\theta}^*_{i2j} = \dfrac{d^*_{ij}}{r_i - n^*_{ij}}$, $i = 1, 2$, $j = 1, 2, \ldots, B$.

Step 5. Arrange $\left\{\widehat{\theta}^*_{ikj} : j = 1, 2, \ldots, B\right\}$ in ascending order to get $\left\{\widehat{\theta}^{*[1]}_{ik} < \ldots < \widehat{\theta}^{*[B]}_{ik}\right\}$, $i = 1, 2$, $k = 1, 2$.

Step 6. A two-sided $100(1 - \alpha)\%$ BCI for θ_{ik} is $\left(\widehat{\theta}^{*[B\frac{\alpha}{2}]}_{ik}, \widehat{\theta}^{*[B(1-\frac{\alpha}{2})]}_{ik}\right)$, $i = 1, 2$, $k = 1, 2$, where $[x]$ denotes the largest integer less than or equal to x.

5 Optimality Criteria and Optimal Test Plans

In this section, we consider the optimal step-stress plans, which optimize different optimality criteria with respect to r_1, when other quantities are assumed to pre-fixed. Here we consider two optimal criteria, which are based on the Fisher information matrix presented below. Note that for given r, r_1 can take values from the finite set $\{2, 3, \ldots, r - 2\}$.

Remark 3. For multiple step-stress experiment designing with k stress levels, s_1, s_2, $\ldots$, s_k, one important issue is to determine the middle stress levels, s_2, $\ldots$, s_{k-1}, since in general the lowest, s_1, and highest, s_k, stress levels can be specified by the experimenter. We do not pursue it in this article and further study is necessary in this direction.

5.1 Fisher Information Matrix

Note that under some regularity conditions the Fisher information matrix for the unknown parameters can be computed by taking expectation of the negative of the second order partial derivatives of log-likelihood function $\log L(\theta_{11},\, \theta_{12},\, \theta_{21},\, \theta_{22})$. Now for $i = 1,\, 2$

$$\frac{\partial}{\partial \theta_{i1}} \log L(\theta_{11},\, \theta_{12},\, \theta_{21},\, \theta_{22}) = -\frac{n_i}{\theta_{i1}} + \frac{D_1}{\theta_{i1}^2},$$
$$\frac{\partial}{\partial \theta_{i2}} \log L(\theta_{11},\, \theta_{12},\, \theta_{21},\, \theta_{22}) = -\frac{r_i - n_i}{\theta_{i2}} + \frac{D_2}{\theta_{i2}^2},$$

and hence

$$\frac{\partial^2}{\partial \theta_{i1}^2} \log L(\theta_{11},\, \theta_{12},\, \theta_{21},\, \theta_{22}) = \frac{n_i}{\theta_{i1}^2} - \frac{2D_1}{\theta_{i1}^3},$$
$$\frac{\partial^2}{\partial \theta_{i2}^2} \log L(\theta_{11},\, \theta_{12},\, \theta_{21},\, \theta_{22}) = \frac{r_i - n_i}{\theta_{i2}^2} - \frac{2D_2}{\theta_{i2}^3},$$
$$\frac{\partial^2}{\partial \theta_{ij}\, \partial \theta_{i'j'}} \log L(\theta_{11},\, \theta_{12},\, \theta_{21},\, \theta_{22}) = 0 \qquad \text{for } i \neq i' \text{ or } j \neq j'.$$

Using Theorems 1, 2, and 3, for $i = 1,\, 2,\, i' = 1,\, 2,\, j = 1,\, 2,\, j' = 1,\, 2,\, i \neq i'$, and $j \neq j'$

$$-E\left(\frac{\partial^2}{\partial \theta_{i1}^2} \log L(\theta_{11},\, \theta_{12},\, \theta_{21},\, \theta_{22})\right) = \frac{r_i p_i}{\theta_{i1}^2},$$
$$-E\left(\frac{\partial^2}{\partial \theta_{i2}^2} \log L(\theta_{11},\, \theta_{12},\, \theta_{21},\, \theta_{22})\right) = \frac{r_i(1-p_i)}{\theta_{i2}^2},$$
$$-E\left(\frac{\partial^2}{\partial \theta_{ij}\, \partial \theta_{i'j'}} \log L(\theta_{11},\, \theta_{12},\, \theta_{21},\, \theta_{22})\right) = 0.$$

Hence the Fisher information matrix of $\theta_{11},\, \theta_{12},\, \theta_{21},\, \theta_{22}$ is given by

$$\boldsymbol{I}(\theta_{11},\, \theta_{12},\, \theta_{21},\, \theta_{22}) = \text{Diag}\left(\frac{r_1 p_1}{\theta_{11}^2},\, \frac{r_1(1-p_1)}{\theta_{12}^2},\, \frac{r_2 p_2}{\theta_{21}^2},\, \frac{r_2(1-p_2)}{\theta_{22}^2}\right).$$

5.2 D-optimality

This optimality criterion is based on the determinant of the Fisher information matrix. Note that the volume of the joint confidence ellipsoid of $(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})$ is inversely proportional to the positive square root of the determinant of the Fisher information matrix $\boldsymbol{I}(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})$ at a fixed confidence level. Thus our objective function for D-optimality is given by

$$\phi_{\mathrm{D}}(r_1) = |\boldsymbol{I}(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})|^{\frac{1}{2}} = \frac{r_1(r - r_1)}{(\theta_{11} + \theta_{12})\,(\theta_{21} + \theta_{22})\,\sqrt{\theta_{11}\theta_{12}\theta_{21}\theta_{22}}}.$$

The D-optimal r_1, say $r_{1\mathrm{D}}$, can be found by maximizing the above objective function. Note that

$$\phi_{\mathrm{D}}(r_1) \lesseqgtr \phi_{\mathrm{D}}(r_1 + 1) \Leftrightarrow r_1 \lesseqgtr \frac{r-1}{2}.$$

Hence the maximum attains at $r_{1\mathrm{D}} = \frac{r}{2}$ for even r, and at $r_{1\mathrm{D}} = \frac{r-1}{2}$ or $\frac{r+1}{2}$ for odd r. Clearly, $r_{1\mathrm{D}}$ does not depend on the parameter values.

5.3 A-optimality

Our next optimality criterion is based on the sum of asymptotic variances of the MLEs of unknown parameters, *i.e.*, it is the sum of diagonal elements of inverse of Fisher information matrix $\boldsymbol{I}(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})$. Thus the objective function of A-optimal criterion is given by

$$\phi_{\mathrm{A}}(r_1) = \frac{\theta_{11}^2}{r_1 p_1} + \frac{\theta_{12}^2}{r_1(1-p_1)} + \frac{\theta_{21}^2}{r_2 p_2} + \frac{\theta_{22}^2}{r_2(1-p_2)} = \frac{\theta_{11}\theta_{12}}{r_1} + \frac{\theta_{21}\theta_{22}}{r - r_1}.$$

A-optimal r_1, say $r_{1\mathrm{A}}$, can be found by minimizing $\phi_{\mathrm{A}}(\cdot)$ with respect to r_1. As it is a discrete optimization problem, one can compute the values of $\phi_{\mathrm{A}}(r_1)$ for all possible values of $r_1 \in \{2, 3, \ldots, r-2\}$ and choose that r_1 which minimizes $\phi_{\mathrm{A}}(r_1)$. Note that $r_{1\mathrm{A}}$ depends on r, θ_{11}, θ_{12}, θ_{21}, and θ_{22}. We provide optimal values of r_1 in the Table 1 for some values of

r, θ_{11}, θ_{12}, θ_{21}, and θ_{22}.

Table 1: The optimal choice of r_1 for different values of θ_{11}, θ_{12}, θ_{21}, and θ_{22}.

(a) $\theta_{11} = 1.0$, $\theta_{12} = 1.100$, $\theta_{21} = 0.50$, $\theta_{22} = 0.550$

r	$r_{1\mathrm{D}}$	$\phi_{\mathrm{D}}(r_{1\mathrm{D}})$	$r_{1\mathrm{A}}$	$\phi_{\mathrm{A}}(r_{1\mathrm{A}})$
5	2	1.2128	3	0.5042
10	5	1.2128	7	0.2488
15	7	1.2128	10	0.1650
20	10	1.2128	13	0.1239
25	12	1.2128	17	0.0991
30	15	1.2128	20	0.0825
35	17	1.2128	23	0.0707
40	20	1.2128	27	0.0619
45	22	1.2128	30	0.0550
50	25	1.2128	33	0.0495

(b) $\theta_{11} = 1.0$, $\theta_{12} = 1.250$, $\theta_{21} = 0.50$, $\theta_{22} = 0.675$

r	$r_{1\mathrm{D}}$	$\phi_{\mathrm{D}}(r_{1\mathrm{D}})$	$r_{1\mathrm{A}}$	$\phi_{\mathrm{A}}(r_{1\mathrm{A}})$
5	2	1.7172	3	0.5854
10	5	1.7172	7	0.2911
15	7	1.7172	10	0.1925
20	10	1.7172	13	0.1444
25	12	1.7172	16	0.1156
30	15	1.7172	20	0.0963
35	17	1.7172	23	0.0825
40	20	1.7172	26	0.0722
45	22	1.7172	30	0.0642
50	25	1.7172	33	0.0577

6 Simulation Study and Data Analysis

6.1 Simulation Study

In this section we consider numerical studies to judge the performance of the MLEs of the unknown parameters and associated ACIs and BCIs. We consider two sets of values for the parameters (θ_{11}, θ_{12}, θ_{21}, θ_{22}), *viz.*, (1.0, 1.1, 0.5, 0.55) and (1.0, 1.25, 0.5, 0.675). As the distributions of $\widehat{\theta}_{i1}$ and $\widehat{\theta}_{i2}$ depend on the ratio of θ_{i1} to θ_{i2}, $i = 1, 2$, we fix θ_{11} and θ_{21} and change the values of θ_{12} and θ_{22}. We consider 20, 30, and 40 as values of n. For each value of n, r is taken to be $[3n/4]$ and n. For each value of r, we take r_1 to be $[5r/11]$, $[r/2]$, and $[6r/11]$. Based on 5000 replication, we compute the average estimates (AE) and mean squared errors (MSE) of the MLEs of the unknown model parameters (see Tables 2–9). The coverage percentages (CP) and average lengths (AL) of the ACI and BCI are also reported in these tables. We take $B = 8000$ for BCI. We notice that the second equation in (4) does not remain feasible for $\alpha = 0.1$ or 0.05, when number of failure due to the cause j at the stress level s_i is one and hence the procedure proposed in Section 4 for construction of the ACI does not work in this case. We discard those data for which the number of failure due to the cause j at the stress level s_i is one for computation of CPs and ALs of θ_{ij}, and the

Table 2: The AEs and MSEs of the MLEs of θ_{11} and the ALs and CPs of the associated ACIs and BCIs with $\theta_{11} = 1.0$, $\theta_{12} = 1.10$, $\theta_{21} = 0.5$, and $\theta_{22} = 0.55$.

					90%				95%				
					ACI		BCI		ACI		BCI		
n	r_1	r_2	AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL	%
20	7	9	1.1914	0.7155	92.94	4.6858	90.58	2.5237	96.82	11.0326	95.24	3.2053	4.45
	8	8	1.1513	0.5939	92.32	3.8477	89.74	2.4057	96.70	7.6388	94.58	3.0843	2.27
	9	7	1.1413	0.5233	91.18	3.3393	89.08	2.3106	96.14	5.9585	94.26	2.9874	1.69
	9	11	1.1485	0.5494	91.78	3.3637	89.92	2.3381	96.44	6.0042	94.92	3.0261	1.38
	10	10	1.1313	0.4580	91.80	2.8070	89.40	2.2079	96.14	4.5072	94.46	2.8797	0.79
	11	9	1.1047	0.3566	90.68	2.5659	89.52	2.0874	95.66	3.9598	94.08	2.7278	0.30
30	10	12	1.1158	0.4161	90.86	2.8295	89.10	2.1928	95.70	4.5755	94.06	2.8569	0.66
	11	11	1.1175	0.3958	90.50	2.5005	88.92	2.0958	95.20	3.7950	93.74	2.7459	0.34
	12	10	1.1044	0.3307	90.60	2.2818	89.12	1.9826	95.76	3.3409	94.16	2.5866	0.22
	13	17	1.0878	0.2963	91.40	2.0044	89.52	1.8395	95.74	2.8081	94.58	2.4062	0.14
	15	15	1.0874	0.2499	90.20	1.7157	89.00	1.6642	95.20	2.2819	94.04	2.1612	0.06
	17	13	1.0731	0.1793	90.36	1.5065	89.54	1.4979	95.48	1.9426	94.34	1.9259	0.00
40	14	16	1.0785	0.2538	90.74	1.8346	89.48	1.7209	95.62	2.5059	94.54	2.2413	0.04
	15	15	1.0874	0.2499	90.20	1.7157	89.00	1.6642	95.20	2.2819	94.04	2.1612	0.06
	16	14	1.0686	0.1822	90.52	1.5673	89.00	1.5462	95.24	2.0391	94.12	2.0009	0.00
	18	22	1.0616	0.1478	90.60	1.3992	89.58	1.4058	95.38	1.7764	94.62	1.8017	0.00
	20	20	1.0457	0.1287	90.28	1.2687	89.36	1.2715	95.46	1.5947	94.50	1.6108	0.00
	22	18	1.0444	0.1114	91.30	1.1879	90.14	1.1867	95.66	1.4841	95.10	1.4913	0.00

Table 3: The AEs and MSEs of the MLEs of θ_{11} and the ALs and CPs of the associated ACIs and BCIs with $\theta_{11} = 1.0$, $\theta_{12} = 1.25$, $\theta_{21} = 0.5$, and $\theta_{22} = 0.675$.

					90%				95%				
					ACI		BCI		ACI		BCI		
n	r_1	r_2	AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL	%
20	7	9	1.1771	0.6390	92.68	4.3734	91.18	2.5012	96.96	9.8340	95.28	3.1926	3.53
	8	8	1.1539	0.5753	91.76	3.5051	89.66	2.3659	96.38	6.5936	94.46	3.0505	2.15
	9	7	1.1249	0.4438	91.20	2.9776	90.18	2.2204	96.10	5.0044	94.44	2.8825	1.94
	9	11	1.1457	0.4913	90.86	3.1672	89.32	2.2667	95.94	5.4974	94.48	2.9377	0.93
	10	10	1.1158	0.3833	90.70	2.5998	89.78	2.1015	95.56	4.0481	94.16	2.7456	0.36
	11	9	1.1120	0.3662	90.88	2.3369	89.68	2.0022	95.76	3.4504	94.42	2.6137	0.28
30	10	12	1.1102	0.3364	91.12	2.6086	90.08	2.1010	96.28	4.0659	94.72	2.7422	0.34
	11	11	1.1104	0.3503	90.80	2.3129	89.02	1.9915	95.64	3.4036	94.38	2.6072	0.20
	12	10	1.0892	0.2921	89.68	2.0358	88.20	1.8351	95.16	2.8657	93.58	2.3939	0.26
	13	17	1.0746	0.2283	90.64	1.8260	89.50	1.7183	95.24	2.4691	94.28	2.2371	0.02
	15	15	1.0634	0.1790	90.12	1.5556	88.86	1.5172	95.10	2.0198	93.90	1.9549	0.00
	17	13	1.0507	0.1452	90.38	1.3780	89.10	1.3600	95.34	1.7514	94.14	1.7322	0.00
40	14	16	1.0786	0.2131	91.42	1.6827	90.32	1.6361	95.96	2.2084	95.12	2.1183	0.04
	15	15	1.0634	0.1790	90.12	1.5556	88.86	1.5172	95.10	2.0198	93.90	1.9549	0.00
	16	14	1.0586	0.1569	90.10	1.4565	89.22	1.4318	95.36	1.8655	94.40	1.8329	0.00
	18	22	1.0523	0.1375	90.16	1.3237	88.78	1.3109	95.06	1.6710	94.12	1.6617	0.00
	20	20	1.0568	0.1274	90.20	1.2380	89.52	1.2284	95.46	1.5511	94.64	1.5449	0.00
	22	18	1.0364	0.0994	91.24	1.1270	90.12	1.1127	95.74	1.4021	95.02	1.3867	0.00

Table 4: The AEs and MSEs of the MLEs of θ_{12} and the ALs and CPs of the associated ACIs and BCIs with $\theta_{11} = 1.0$, $\theta_{12} = 1.10$, $\theta_{21} = 0.5$, and $\theta_{22} = 0.55$.

					90%				95%				
					ACI		BCI		ACI		BCI		
n	r_1	r_2	AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL	%
20	7	9	1.3376	0.9609	93.84	5.5800	90.92	2.8227	97.24	13.7197	95.44	3.5565	7.15
	8	8	1.3145	0.8417	93.54	5.0082	90.54	2.7870	97.16	10.7353	95.22	3.5387	4.05
	9	7	1.3159	0.8345	91.74	4.3676	89.18	2.7475	96.54	8.3578	94.58	3.5238	2.97
	9	11	1.2641	0.6800	92.10	4.2301	89.62	2.6563	96.64	8.0539	94.82	3.4148	1.96
	10	10	1.2867	0.7077	91.66	3.7427	89.20	2.6426	96.32	6.5096	94.36	3.4238	1.36
	11	9	1.2305	0.5343	91.00	3.1530	88.96	2.4561	95.70	5.0756	93.72	3.2069	0.79
30	10	12	1.2756	0.6858	91.24	3.7363	88.50	2.6000	96.34	6.5452	93.96	3.3659	1.75
	11	11	1.2529	0.5845	91.34	3.2178	89.18	2.4890	95.92	5.1998	94.62	3.2458	0.85
	12	10	1.2213	0.4367	90.72	2.8560	88.84	2.3452	95.80	4.3982	94.22	3.0770	0.40
	13	17	1.2171	0.4363	90.86	2.5858	89.12	2.2363	95.76	3.8302	94.34	2.9305	0.22
	15	15	1.2149	0.3630	90.38	2.1443	89.02	2.0506	95.44	2.9482	93.96	2.6889	0.10
	17	13	1.1921	0.2776	90.94	1.8467	89.72	1.8209	95.32	2.4521	94.40	2.3685	0.04
40	14	16	1.2102	0.3416	91.34	2.2720	89.84	2.1304	95.82	3.1772	94.78	2.8011	0.12
	15	15	1.2149	0.3630	90.38	2.1443	89.02	2.0506	95.44	2.9482	93.96	2.6889	0.10
	16	14	1.2059	0.3241	90.20	1.9933	88.88	1.9327	95.48	2.6978	94.10	2.5252	0.08
	18	22	1.1827	0.2518	91.14	1.7548	89.66	1.7229	95.44	2.3208	94.40	2.2323	0.02
	20	20	1.1708	0.2062	89.68	1.5509	88.84	1.5775	95.06	1.9812	93.84	2.0281	0.02
	22	18	1.1544	0.1571	90.54	1.4030	89.58	1.4297	95.48	1.7647	94.42	1.8220	0.00

Table 5: The AEs and MSEs of the MLEs of θ_{12} and the ALs and CPs of the associated ACIs and BCIs with $\theta_{11} = 1.0$, $\theta_{12} = 1.25$, $\theta_{21} = 0.5$, and $\theta_{22} = 0.675$.

					90%				95%				
					ACI		BCI		ACI		BCI		
n	r_1	r_2	AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL	%
20	7	9	1.5201	1.2359	94.06	6.5457	90.98	3.1691	97.34	16.5717	96.16	3.9709	9.31
	8	8	1.5137	1.1654	93.16	5.8453	90.32	3.1855	97.20	12.7452	95.10	4.0278	6.10
	9	7	1.4938	1.0874	92.68	5.3947	89.88	3.1869	96.68	10.6579	95.16	4.0663	4.10
	9	11	1.5037	1.1062	92.94	5.3224	90.14	3.1651	96.96	10.5102	94.90	4.0317	3.66
	10	10	1.4800	1.0132	92.08	4.5818	89.20	3.0759	96.44	8.2352	94.26	3.9654	2.48
	11	9	1.4487	0.8691	92.08	4.0665	89.64	2.9602	96.90	6.8594	94.88	3.8433	1.48
30	10	12	1.4780	1.0264	92.06	4.5763	89.48	3.0826	96.52	8.2101	94.02	3.9694	2.38
	11	11	1.4485	0.8909	91.92	4.0293	89.20	2.9536	96.62	6.7701	94.62	3.8376	1.32
	12	10	1.4409	0.8559	91.32	3.7248	89.42	2.8634	96.08	6.0245	94.12	3.7373	0.93
	13	17	1.4162	0.6960	91.34	3.2779	89.48	2.7273	96.02	5.0211	94.32	3.5699	0.60
	15	15	1.3961	0.5572	91.28	2.6965	89.54	2.4831	95.74	3.8430	94.62	3.2601	0.26
	17	13	1.3699	0.3914	90.66	2.3333	89.28	2.2485	95.86	3.1943	94.58	2.9472	0.02
40	14	16	1.3979	0.5879	90.70	2.8601	89.04	2.5956	95.74	4.1408	94.10	3.4203	0.28
	15	15	1.3961	0.5572	91.28	2.6965	89.54	2.4831	95.74	3.8430	94.62	3.2601	0.26
	16	14	1.3942	0.4795	90.86	2.5329	89.28	2.3903	96.00	3.5375	94.70	3.1344	0.14
	18	22	1.3622	0.4046	91.06	2.1441	89.48	2.1303	95.72	2.8654	94.44	2.7814	0.08
	20	20	1.3552	0.3271	89.94	1.9134	88.68	1.9575	95.16	2.4762	94.12	2.5387	0.04
	22	18	1.3455	0.2895	90.30	1.7762	89.34	1.8160	95.28	2.2763	94.40	2.3361	0.00

Table 6: The AEs and MSEs of the MLEs of θ_{21} and the ALs and CPs of the associated ACIs and BCIs with $\theta_{11} = 1.0$, $\theta_{12} = 1.10$, $\theta_{21} = 0.5$, and $\theta_{22} = 0.55$.

					90%				95%				
					ACI		BCI		ACI		BCI		
n	r_1	r_2	AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL	%
20	7	9	0.5755	0.1387	91.70	1.6792	89.58	1.1722	96.58	2.9847	94.56	1.5182	1.61
	8	8	0.5773	0.1380	91.84	1.9662	90.20	1.2146	96.36	3.9245	94.90	1.5594	2.08
	9	7	0.5955	0.1798	92.58	2.3830	90.30	1.2630	96.86	5.6529	95.10	1.6034	4.58
	9	11	0.5480	0.0815	89.96	1.2291	88.34	1.0231	95.78	1.8657	94.10	1.3398	0.28
	10	10	0.5721	0.1176	91.26	1.4569	89.34	1.1223	95.88	2.3714	94.30	1.4597	0.81
	11	9	0.5722	0.1390	91.66	1.6470	89.40	1.1565	96.52	2.9206	94.10	1.4951	1.32
30	10	12	0.5475	0.0735	91.56	1.0984	89.92	0.9719	96.02	1.5861	94.46	1.2758	0.18
	11	11	0.5574	0.0931	91.38	1.2785	89.68	1.0456	96.12	1.9679	94.40	1.3664	0.34
	12	10	0.5610	0.1178	90.40	1.4248	88.56	1.0955	95.36	2.3170	93.52	1.4281	0.62
	13	17	0.5310	0.0418	90.82	0.7428	89.52	0.7307	95.64	0.9616	94.38	0.9385	0.02
	15	15	0.5408	0.0593	90.36	0.8467	88.92	0.8264	95.42	1.1193	94.34	1.0721	0.04
	17	13	0.5400	0.0662	90.34	0.9770	89.10	0.9087	95.58	1.3507	93.78	1.1886	0.08
40	14	16	0.5335	0.0529	90.58	0.7820	89.38	0.7737	95.50	1.0175	93.98	0.9988	0.06
	15	15	0.5408	0.0593	90.36	0.8467	88.92	0.8264	95.42	1.1193	94.34	1.0721	0.04
	16	14	0.5392	0.0592	91.14	0.8990	89.62	0.8632	95.54	1.2073	94.52	1.1252	0.02
	18	22	0.5248	0.0296	90.64	0.5966	89.62	0.5960	94.98	0.7452	94.48	0.7492	0.00
	20	20	0.5292	0.0337	90.56	0.6469	89.50	0.6480	95.56	0.8148	94.58	0.8218	0.00
	22	18	0.5279	0.0385	90.16	0.6975	89.46	0.6945	95.54	0.8895	94.36	0.8877	0.00

Table 7: The AEs and MSEs of the MLEs of θ_{21} and the ALs and CPs of the associated ACIs and BCIs with $\theta_{11} = 1.0$, $\theta_{12} = 1.25$, $\theta_{21} = 0.5$, and $\theta_{22} = 0.675$.

					90%				95%				
					ACI		BCI		ACI		BCI		
n	r_1	r_2	AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL	%
20	7	9	0.5599	0.1037	91.46	1.4332	89.98	1.0860	96.10	2.3730	94.78	1.4125	1.38
	8	8	0.5706	0.1309	91.30	1.7136	89.84	1.1618	96.64	3.1894	94.78	1.4977	1.44
	9	7	0.5942	0.1653	91.64	2.1037	90.40	1.2486	96.48	4.6064	94.84	1.5957	3.72
	9	11	0.5482	0.0750	90.22	1.0862	89.22	0.9580	95.18	1.5542	93.96	1.2471	0.20
	10	10	0.5573	0.0845	91.38	1.2661	90.24	1.0280	96.14	1.9505	95.04	1.3388	0.30
	11	9	0.5602	0.1149	91.06	1.4033	89.14	1.0795	96.14	2.3023	94.44	1.4034	0.77
30	10	12	0.5404	0.0595	90.76	0.9558	89.44	0.8836	95.30	1.3076	94.14	1.1511	0.14
	11	11	0.5461	0.0695	90.64	1.0727	89.30	0.9465	95.72	1.5326	94.52	1.2366	0.10
	12	10	0.5517	0.0815	90.94	1.1975	89.50	1.0134	95.90	1.7839	94.56	1.3239	0.36
	13	17	0.5219	0.0355	90.28	0.6656	89.12	0.6548	95.16	0.8411	94.04	0.8307	0.00
	15	15	0.5339	0.0450	89.92	0.7715	88.70	0.7425	95.56	1.0044	94.14	0.9510	0.00
	17	13	0.5367	0.0544	90.86	0.8873	89.72	0.8268	95.70	1.1976	94.38	1.0715	0.00
40	14	16	0.5320	0.0394	90.60	0.7159	90.04	0.7007	95.36	0.9131	94.58	0.8933	0.02
	15	15	0.5339	0.0450	89.92	0.7715	88.70	0.7425	95.56	1.0044	94.14	0.9510	0.00
	16	14	0.5295	0.0479	89.80	0.7994	88.58	0.7692	95.24	1.0450	93.74	0.9914	0.00
	18	22	0.5169	0.0238	90.22	0.5487	89.46	0.5384	95.38	0.6811	94.26	0.6683	0.00
	20	20	0.5197	0.0273	90.54	0.5899	89.74	0.5799	95.82	0.7365	94.74	0.7251	0.00
	22	18	0.5282	0.0353	89.48	0.6512	88.48	0.6415	95.22	0.8202	93.98	0.8100	0.00

Table 8: The AEs and MSEs of the MLEs of θ_{22} and the ALs and CPs of the associated ACIs and BCIs with $\theta_{11} = 1.0$, $\theta_{12} = 1.10$, $\theta_{21} = 0.5$, and $\theta_{22} = 0.55$.

					90%				95%				
					ACI		BCI		ACI		BCI		
n	r_1	r_2	AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL	%
20	7	9	0.6490	0.1896	93.12	2.1152	90.22	1.3535	97.30	4.0000	95.08	1.7387	2.86
	8	8	0.6592	0.2161	92.78	2.4472	90.14	1.3937	96.70	5.1880	94.70	1.7728	4.20
	9	7	0.6614	0.2243	93.48	2.7867	90.76	1.3953	97.32	6.8729	95.32	1.7578	7.13
	9	11	0.6274	0.1526	91.42	1.6043	89.58	1.2473	96.18	2.5922	94.66	1.6269	0.89
	10	10	0.6402	0.1755	91.14	1.8844	89.30	1.3120	96.06	3.2924	94.10	1.6991	1.34
	11	9	0.6495	0.1872	92.34	2.1912	90.32	1.3720	97.02	4.1862	94.78	1.7592	2.15
30	10	12	0.6209	0.1253	91.08	1.4546	89.18	1.1892	95.82	2.2470	94.46	1.5555	0.48
	11	11	0.6281	0.1509	91.16	1.6338	88.98	1.2459	95.86	2.6659	93.76	1.6253	0.83
	12	10	0.6458	0.1876	91.82	1.8818	89.44	1.3197	96.38	3.2876	94.36	1.7060	1.67
	13	17	0.5976	0.0672	91.02	0.9027	89.70	0.9158	95.54	1.1775	94.56	1.1959	0.06
	15	15	0.6052	0.0832	90.82	1.0638	89.36	1.0161	95.28	1.4618	93.88	1.3334	0.08
	17	13	0.6177	0.1150	91.16	1.3066	89.52	1.1414	95.98	1.9259	94.76	1.4979	0.26
40	14	16	0.5996	0.0775	90.36	0.9832	88.72	0.9617	95.36	1.3228	93.98	1.2576	0.10
	15	15	0.6052	0.0832	90.82	1.0638	89.36	1.0161	95.28	1.4618	93.88	1.3334	0.08
	16	14	0.6106	0.0948	90.44	1.1940	88.86	1.0786	95.78	1.7099	94.22	1.4150	0.08
	18	22	0.5841	0.0419	90.54	0.7149	89.52	0.7284	95.34	0.9010	94.52	0.9279	0.00
	20	20	0.5894	0.0523	89.24	0.7791	88.24	0.7953	94.98	0.9920	93.76	1.0221	0.00
	22	18	0.5921	0.0605	90.52	0.8761	89.32	0.8659	95.46	1.1531	94.38	1.1221	0.00

Table 9: The AEs and MSEs of the MLEs of θ_{22} and the ALs and CPs of the associated ACIs and BCIs with $\theta_{11} = 1.0$, $\theta_{12} = 1.25$, $\theta_{21} = 0.5$, and $\theta_{22} = 0.675$.

					90%				95%				
					ACI		BCI		ACI		BCI		
n	r_1	r_2	AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL	%
20	7	9	0.8096	0.3339	92.54	2.9696	89.46	1.7122	96.90	5.9572	94.58	2.1739	4.92
	8	8	0.8289	0.3806	93.38	3.3131	90.16	1.7233	96.82	7.4081	94.76	2.1658	7.58
	9	7	0.8246	0.3713	93.48	3.7430	90.32	1.7088	97.28	9.7404	95.00	2.1319	10.60
	9	11	0.7985	0.2905	91.26	2.4056	88.38	1.6521	96.24	4.1875	93.80	2.1327	2.08
	10	10	0.8050	0.3158	92.30	2.7084	89.68	1.6932	96.64	5.0474	94.68	2.1715	2.53
	11	9	0.8104	0.3287	93.34	2.9771	90.28	1.7179	97.36	5.9812	95.26	2.1811	4.25
30	10	12	0.7924	0.2655	91.32	2.1814	89.12	1.6118	96.06	3.6145	94.06	2.0970	1.28
	11	11	0.8013	0.2796	91.66	2.3755	89.44	1.6626	96.08	4.1045	94.12	2.1527	2.00
	12	10	0.8097	0.3132	92.70	2.6865	89.80	1.7055	96.84	4.9757	94.82	2.1874	2.97
	13	17	0.7494	0.1467	90.16	1.3342	88.88	1.2775	95.42	1.8539	94.00	1.6755	0.18
	15	15	0.7594	0.1723	90.76	1.5569	89.02	1.3978	95.92	2.2687	94.04	1.8374	0.32
	17	13	0.7766	0.2164	91.10	1.8966	88.92	1.5440	96.02	2.9572	94.06	2.0200	0.71
40	14	16	0.7474	0.1528	91.32	1.3547	89.76	1.3143	95.60	1.8740	94.20	1.7300	0.22
	15	15	0.7594	0.1723	90.76	1.5569	89.02	1.3978	95.92	2.2687	94.04	1.8374	0.32
	16	14	0.7672	0.1873	91.66	1.7857	89.88	1.4725	95.78	2.7394	94.38	1.9278	0.42
	18	22	0.7323	0.0867	90.16	0.9985	89.14	1.0325	95.40	1.2852	94.50	1.3365	0.02
	20	20	0.7355	0.1065	90.36	1.0984	89.44	1.1113	95.48	1.4489	94.50	1.4490	0.04
	22	18	0.7449	0.1195	90.12	1.2549	89.02	1.2285	95.18	1.7102	93.98	1.6095	0.04

percentages of this type of data are provided in the last column of Tables 2–9. Clearly these percentages for the first stress level depend on the distance between θ_{11} and θ_{12}. As the distance between θ_{11} and θ_{12} increases, percentage of data having only one failure due to the cause with larger mean life increases. Similar relationship also holds for the second stress level.

The following points are quite clear from the Tables 2–9. As n increases, the MSEs of estimators of all the unknown parameters decrease. For fixed n, the MSEs of estimators of model parameters decrease as r increases. For fixed n and r, the MSEs of the MLEs of θ_{11} and θ_{12} decrease and that of the MLEs of θ_{21} and θ_{22} increase with the increase in r_1. For $\theta_{11} < \theta_{12}$, the MSE of the MLE of θ_{11} decreases and that of θ_{12} increases as $|\theta_{11} - \theta_{12}|$ increases keeping n and r_1 fixed. Similar trend is also noticed for θ_{21} and θ_{22} for fixed n and r_2. The CPs of the ACIs and BCIs are quite close to nominal level, though the ALs of BCIs are smaller than that of ACIs specially for small values of n. As n increases, the difference between the ALs of the ACIs and BCIs decreases. However, as we discussed, the ACI of θ_{ij}, $i = 1, 2$, $j = 1, 2$, has an issue when number of failure is very small due to the cause j at the stress level s_i. Hence, we recommend to use BCI over ACI. However, we have noticed that as n increases, the percentage of discarded samples due to the infeasibility of (4) decreases, specially for relatively large values of r_1.

6.2 Data Analysis

In this subsection we provide the analysis of two data sets to illustrate the methods described in Sections 2 and 4. Both the data are artificially generated from (1) with $n = 30$, $r_1 = 10$, $r_2 = 13$, $\theta_{11} = 1$, $\theta_{12} = 1.25$, $\theta_{21} = 0.5$, and $\theta_{22} = 0.675$. The data are given in Tables 10 and 12.

Table 10: The data for illustrative example 1.

First stress level	0.024519, 0	0.046106, 0	0.052244, 0	0.096887, 0	0.117613, 1	0.181907, 0	0.183071, 1
	0.201780, 1	0.247922, 1	0.293073, 0				
Second stress level	0.305320, 1	0.310835, 1	0.313429, 0	0.325594, 1	0.338804, 1	0.362189, 1	0.368959, 0
	0.380403, 1	0.422346, 0	0.425203, 1	0.544151, 0	0.599206, 0	0.616952, 1	

Table 11: The ACI and BCI for illustrative example 1.

	ACI				BCI			
	90%		95%		90%		95%	
Parameter	LL	UL	LL	UL	LL	UL	LL	UL
θ_{11}	0.666115	2.701967	0.597058	3.232235	0.601998	2.488081	0.524194	2.892671
θ_{12}	0.876948	5.092153	0.770504	6.605372	0.809575	4.998344	0.706583	6.673072
θ_{21}	0.388649	1.804309	0.345897	2.223212	0.375779	1.863981	0.333322	2.370978
θ_{22}	0.279012	0.923804	0.253572	1.069155	0.254216	0.861798	0.224597	0.988965

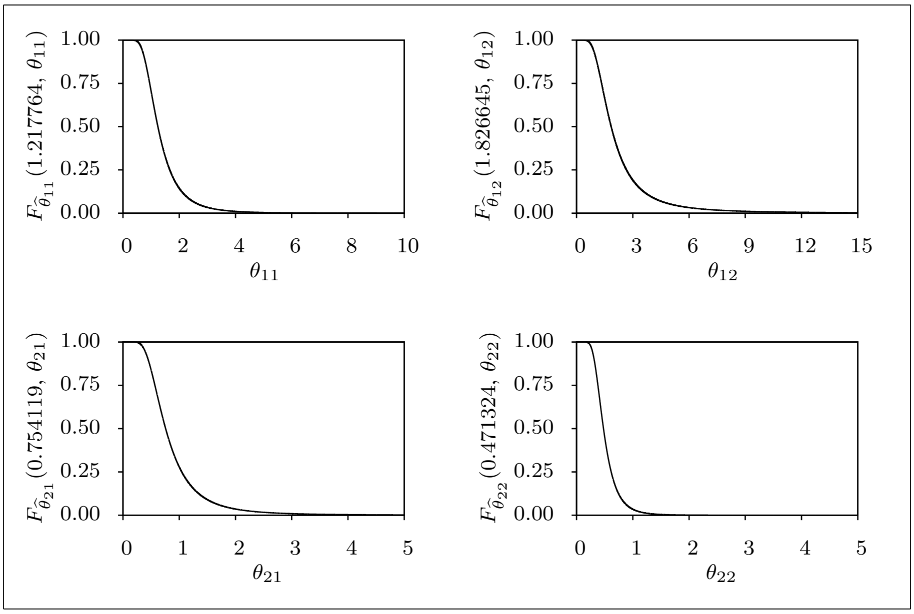

Figure 1: The plots of CDFs of $\widehat{\theta}_{ij}$ as a function of θ_{ij} for data in Table 10.

Illustrative Example 1

Here we provide the analysis of the data given in Table 10. For this data, number of failures due to the first cause at the first and second stress levels are 6 and 5, respectively and that due to second cause are 4 and 8, respectively. The MLEs of θ_{11}, θ_{12}, θ_{21}, and θ_{22} can be obtained using (2) and are 1.217764, 1.826645, 0.754119, and 0.471324, respectively. Plots of $\big(\theta_{11},\, F_{\widehat{\theta}_{11}}(1.217764,\, \theta_{11})\big)$, $\big(\theta_{12},\, F_{\widehat{\theta}_{12}}(1.826645,\, \theta_{12})\big)$, $\big(\theta_{21},\, F_{\widehat{\theta}_{21}}(0.754119,\, \theta_{21})\big)$, and $\big(\theta_{22},\, F_{\widehat{\theta}_{22}}(0.471324,\, \theta_{22})\big)$ are given in Figure 1. Clearly, (4) given in Section 4 are feasible for 90% and 95% CIs. We construct 90% and 95% ACIs, reported in Table 11, for all the unknown parameters by solving (4) using bisection method. One-dimensional bisection method needs two values such that only one root of the concerned equation lies in between them. We find out these values from Figure 1. The BCI can be constructed following the method described in Section 4. We compute 90% and 95% BCIs, and they are reported in

Table 11. We have noticed that the ACI and BCI of each parameter are quite close for a fixed level of confidence. The lengths of the BCIs are marginally smaller than that of the ACIs for θ_{11}, θ_{12}, and θ_{22}, while the relation is reversed for θ_{21} for the data provided in Table 10.

Table 12: The data for illustrative example 2.

First stress level	0.026226, 1	0.069863, 1	0.095057, 1	0.096341, 1	0.104098, 1	0.175074, 0	0.194085, 1
	0.226281, 1	0.227065, 1	0.274758, 1				
Second stress level	0.278594, 0	0.291634, 0	0.296244, 0	0.314911, 0	0.321721, 1	0.327938, 0	0.362588, 0
	0.396857, 0	0.411424, 0	0.423408, 1	0.447501, 0	0.448236, 0	0.456391, 1	

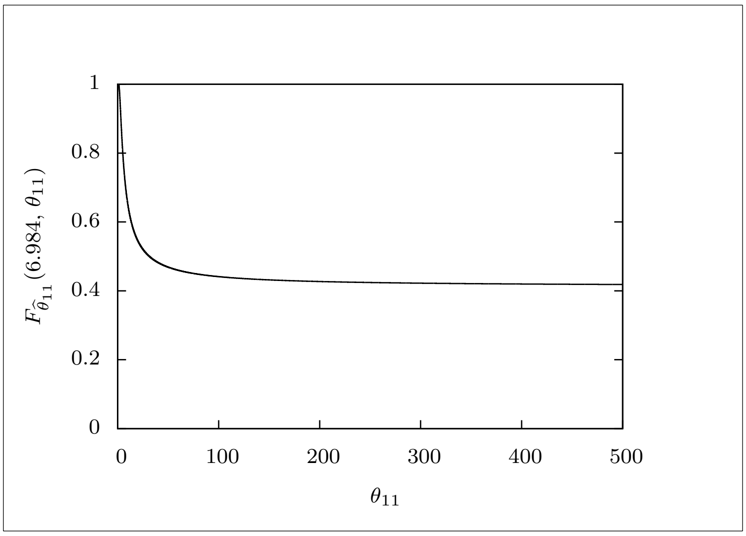

Figure 2: The plot of CDF of $\widehat{\theta}_{11}$ as a function of θ_{11} for data in Table 12.

Illustrative Example 2

Here we provide a data in Table 12 with number of failure due to the first cause at first stress level is one. The MLEs of the parameters for this data are $\widehat{\theta}_{11} = 6.984$, $\widehat{\theta}_{12} = 0.776$, $\widehat{\theta}_{21} = 0.248$, and $\widehat{\theta}_{22} = 0.826$. As there is only one failure due to the first cause of failure at the first stress level, the MLE of θ_{11} is unusually larger than its true value. Plot of $F_{\widehat{\theta}_{11}}(6.984, \theta_{11})$ as a function of θ_{11} is provided in Figure 2, which depicts that as θ_{11} increases $F_{\widehat{\theta}_{11}}(6.984, \theta_{11})$ stabilizes near 0.42. Hence, the second equation in (4) will be feasible for $\alpha > 0.84$, which implies we can have at the maximum 16% two-sided confidence interval for θ_{11} for this data using the approximate method of confidence interval. If one-sided confidence interval of the form $(0, \theta_{11\mathrm{U}})$ is considered, the maximum level of significance that can be achieved using the approximate method for CI is close to 58%.

7 Conclusion

In this article we consider a simple SSLT with random step changing time. We also assume that in each stress levels there are two independent competing risks acting simultaneously on the units under consideration. We assume that the lifetimes are distributed exponentially in the presence of only one cause. It is further assumed that the assumptions of the CEM hold for the lifetimes for each cause of failure in the absence of other cause. We have obtained the MLEs of the model parameters and their exact conditional distributions. Based on the conditional distributions of the MLEs, we proposed the ACI and BCI. We discuss two optimality criteria and the optimal tests under those optimal criteria. We conduct an extensive simulation study to judge the performance of the procedures proposed in this article. We have noticed that approximate method of confidence interval does not work when number of failure at the associated stress level due to associated risk is very few. However, the BCI works quite well for all the cases and hence we recommend to use the BCI over ACI. Further study is needed towards the construction of confidence interval of the model parameters in case of very few failures at the associated stress level due to associated risk. Note that in this paper we have considered the simple step-stress model. It will be of interest to develop statistical inferences of the unknown parameters in case of multiple step-stress model under this framework. More work is needed in this direction.

Acknowledgements

The authors would like to thank the referees and the associate editor for their very constructive comments.

A Appendix

Lemma 1. Suppose that X is a random variable having a gamma distribution with PDF $f_{\mathrm{G}}(\cdot;\, \alpha,\, \beta)$ as given in (3). Then for $\omega < \beta$, the MGF of X is given by

$$E\left(e^{\omega X}\right) = \left(1 - \frac{\omega}{\beta}\right)^{-\alpha}.$$

Proof: Proof is simple and hence it is omitted.

Proof of the Theorem 1: Joint PDF of $\boldsymbol{T}_{1,r_1}$ is given by

$$\begin{aligned} f_{\boldsymbol{T}_{1,r_1}}\left(\boldsymbol{t}_{1,r_1}\right) &= \sum f_{\boldsymbol{T}_{1,r_1},\, \boldsymbol{\Delta}_{1,r_1}}\left(\boldsymbol{t}_{1,r_1},\, \boldsymbol{\delta}_{1,r_1}\right) \\ &= c_{1,r_1,n}\left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right)^{r_1} e^{-\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{21}}\right)D_1} \qquad \text{for } 0 < t_{1:n} < \ldots < t_{r_1:n} < \infty, \end{aligned}$$

where the sum is taken over all the possible values of $\delta_{i:n}$ for $i = 1,\, 2, \ldots,\, r_1$. Hence for $\omega < \dfrac{1}{\theta_{11}} + \dfrac{1}{\theta_{12}}$

$$\begin{aligned} E\left(e^{\omega D_1}\right) &= \int_0^\infty \cdots \int_{t_{r_1-1:n}}^\infty c_{1,r_1,n}\left(\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}\right)^{r_1} e^{-\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}-\omega\right)D_1} dt_{r_1:n} \ldots dt_{1:n} \\ &= \left(1 - \frac{\omega}{\frac{1}{\theta_{11}} + \frac{1}{\theta_{12}}}\right)^{-r_1}. \end{aligned}$$

The above expression of the MGF of D_1 along with Lemma 1 proves the theorem. □

Proof of the Theorem 2: The conditional joint PDF of $\boldsymbol{T}_{r_1+1,r}$ conditioning on $T_{r_1:n}$ is given by

$$f_{\boldsymbol{T}_{r_1+1,r}\,|\,T_{r_1:n}}(\boldsymbol{t}_{r_1+1,r}\,|\,t_{r_1:n})$$

$$= \frac{\sum \int_0^{t_{r_1:n}} \cdots \int_0^{t_{2:n}} f_{\boldsymbol{T}_{1,r},\, \boldsymbol{\Delta}_{1,r}}\left(\boldsymbol{t}_{1,r},\, \boldsymbol{\delta}_{1,r}\right) dt_{1:n} \ldots dt_{r_1-1:n}}{\sum \int_0^{t_{r_1:n}} \cdots \int_0^{t_{2:n}} \int_{t_{r_1:n}}^\infty \cdots \int_{t_{r-1:n}}^\infty f_{\boldsymbol{T}_{1,r},\, \boldsymbol{\Delta}_{1,r}}\left(\boldsymbol{t}_{1,r},\, \boldsymbol{\delta}_{1,r}\right) dt_{r:n} \ldots t_{r_1+1:n} dt_{1:n} \ldots dt_{r_1-1:n}}$$

$$= c_{r_1+1,r,n} \left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)^{r_2} e^{-\left(\frac{1}{\theta_{21}}+\frac{1}{\theta_{22}}\right)D_2} \qquad \text{for } t_{r_1:n} < t_{r_1+1:n} < \ldots < t_{r:n} < \infty,$$

where the sum is over all possible values of $\delta_{i:n}$ for $i = 1, 2, \ldots, r$. Now for $\omega < \left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)$,

$$E(e^{\omega D_2} \,|\, t_{r_1:n}) = c_{r_1+1,r,n} \left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)^{r_2} \int_{t_{r_1:n}}^{\infty} \cdots \int_{t_{r-1:n}}^{\infty} e^{-\left(\frac{1}{\theta_{21}}+\frac{1}{\theta_{22}}-\omega\right)D_2} dt_{r:n} \ldots dt_{r_1+1:n}$$

$$= \left(1 - \frac{\omega}{\left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)}\right)^{-r_2}.$$

Hence

$$E(e^{\omega D_2}) = \int_0^{\infty} E\left(e^{\omega\,\widehat{\theta}_{21}} \,|\, t_{r_1:n}\right) f_{T_{r_1:n}}(t_{r_1:n}) dt_{r_1:n} = \left(1 - \frac{\omega}{\left(\frac{1}{\theta_{21}} + \frac{1}{\theta_{22}}\right)}\right)^{-r_2},$$

which along with the Lemma 1, completes the proof of Theorem 2. □

Proof of the Theorem 3: Using (1), for $\boldsymbol{n} \in \mathcal{N}$,

$$P(\boldsymbol{N} = \boldsymbol{n}) = \sum\nolimits^{1} \sum\nolimits^{2} \int_0^{\infty} \cdots \int_{t_{r-1:n}}^{\infty} f_{\boldsymbol{T}_{1,r},\,\boldsymbol{\Delta}_{1,r}}(\boldsymbol{t}_{1,r},\, \boldsymbol{\delta}_{1,r}) dt_{r:n} \ldots dt_{1:n}$$

$$= \binom{r_1}{n_1}\binom{r_2}{n_2} c_{1,r}\, \lambda_{11}^{n_1}\, \lambda_{12}^{r_1-n_1}\, \lambda_{21}^{n_2}\, \lambda_{22}^{r_2-n_2}$$

$$\times \int_0^{\infty} \cdots \int_{t_{r-1:n}}^{\infty} e^{-(\lambda_{11}+\lambda_{12})D_1 - (\lambda_{21}+\lambda_{22})D_2} dt_{r:n} \ldots dt_{1:n}$$

$$= \binom{r_1}{n_1} p_1^{n_1} \left(1 - p_1\right)^{r_1-n_1} \times \binom{r_2}{n_2} p_2^{n_2} \left(1 - p_2\right)^{r_2-n_2},$$

where $\sum^1$ and $\sum^2$ imply the summations over all possible arrangement of $\delta_{i:n}$ such that $\sum_{i=1}^{r_1} \delta_{i:n} = n_1$ and $\sum_{i=r_1+1}^{r} \delta_{i:n} = n_2$, respectively. □

Proof of the Theorem 4: Note that for $i = 1, 2$ and $j = 1, 2$

$$E\left(e^{\omega\,\widehat{\theta}_{ij}} | \boldsymbol{N} \in \mathcal{N}\right) = \sum_{\boldsymbol{n} \in \mathcal{N}} E\left(e^{\omega\,\widehat{\theta}_{ij}} | \boldsymbol{N} = \boldsymbol{n}\right) \times P(\boldsymbol{N} = \boldsymbol{n} | \boldsymbol{N} \in \mathcal{N}). \tag{5}$$

Now for $\boldsymbol{n} \in \mathcal{N}$ and using Theorem 3,

$$\begin{aligned}
P(\boldsymbol{N}=\boldsymbol{n}|\boldsymbol{N}\in\mathcal{N}) =& \frac{P(\boldsymbol{N}=\boldsymbol{n})}{P(\boldsymbol{N}\in\mathcal{N})} \\
=& \frac{\binom{r_1}{n_1}p_1^{n_1}\left(1-p_1\right)^{r_1-n_1}\times\binom{r_2}{n_2}p_2^{n_2}\left(1-p_2\right)^{r_2-n_2}}{\sum_{\boldsymbol{n}\in\mathcal{N}}\binom{r_1}{n_1}p_1^{n_1}\left(1-p_1\right)^{r_1-n_1}\times\binom{r_2}{n_2}p_2^{n_2}\left(1-p_2\right)^{r_2-n_2}} \\
=& \frac{\binom{r_1}{n_1}p_1^{n_1}\left(1-p_1\right)^{r_1-n_1}\times\binom{r_2}{n_2}p_2^{n_2}\left(1-p_2\right)^{r_2-n_2}}{\left\{1-p_1^{r_1}-(1-p_1)^{r_1}\right\}\left\{1-p_2^{r_2}-(1-p_2)^{r_2}\right\}}.
\end{aligned} \tag{6}$$

Now using Theorem 1 or 2, for $k = 1, 2, \ldots, r_i - 1$, $i = 1, 2$, and $j = 1, 2$,

$$E\left(e^{\omega\widehat{\theta}_{ij}}|N_i=k\right) = \left(1-\frac{\omega}{n_{ijk}\left(\frac{1}{\theta_{i1}}+\frac{1}{\theta_{i2}}\right)}\right)^{-r_i} \quad \text{if } \omega < n_{ijk}\left(\frac{1}{\theta_{i1}}+\frac{1}{\theta_{i2}}\right).$$

Hence, using (5) and (6), the MGF of $\widehat{\theta}_{11}$ conditioning on $\boldsymbol{N} \in \mathcal{N}$ is given by

$$E\left(e^{\omega\widehat{\theta}_{ij}}|\boldsymbol{N}\in\mathcal{N}\right) = \frac{1}{1-p_i^{r_i}-(1-p_i)^{r_i}}\sum_{k=1}^{r_i-1}\binom{r_i}{k}p_i^k\left(1-p_i\right)^{r_i-k}\left(1-\frac{\omega}{n_{ijk}\left(\frac{1}{\theta_{i1}}+\frac{1}{\theta_{i2}}\right)}\right)^{-r_i},$$

if $\omega < \dfrac{1}{\theta_{i1}} + \dfrac{1}{\theta_{i2}}$. Now using Lemma 1, the proof of Theorem 4 is straight forward.

Proof of the Theorem 5: Note that for $i = 1, 2$ and $j = 1, 2$,

$$1-F_{\widehat{\theta}_{ij}}(x,\theta_{ij}) = \sum_{k=1}^{r_i-1}\mathrm{P}_{\theta_{ij}}\left(\widehat{\theta}_{ij}>x|N_i=k\right)\mathrm{P}_{\theta_{ij}}\left(N_i=k|1\le N_i\le r_i-1\right), \tag{7}$$

which is in the form of (1) given in Balakrishnan and Iliopoulos [9]. Hence we will prove that M1, M2, and M3 of Lemma 1 in Balakrishnan and Iliopoulos [9] hold for (7) to prove stochastic monotonicity of $\widehat{\theta}_{ij}$ in θ_{ij}.

(M1) Note that

$$\mathrm{P}_{\theta_{ij}}\left(\widehat{\theta}_{ij}>x|N_i=k\right) = \int_0^x f_{\mathrm{G}}\left(t;\, r_i,\, n_{ijk}\left(\frac{1}{\theta_{i1}}+\frac{1}{\theta_{i2}}\right)\right)dt.$$

Now for $\theta_{ij}^{'} > \theta_{ij}$, $\dfrac{f_{\mathrm{G}}\left(t;\, r_i,\, n_{ijk}\left(\frac{1}{\theta_{ij}^{'}} + \frac{1}{\theta_{ij'}}\right)\right)}{f_{\mathrm{G}}\left(t;\, r_i,\, n_{ijk}\left(\frac{1}{\theta_{ij}} + \frac{1}{\theta_{ij'}}\right)\right)}$ is an increasing function of t, where $j^{'} \neq j$ and $j^{'} = 1,\, 2$. Hence $\mathrm{P}_{\theta_{ij}}\left(\widehat{\theta}_{ij} > x | N_i = k\right)$ is an increasing function of θ_{ij}.

(M2) The distribution of $\widehat{\theta}_{ij}|N_i = k$ is same as $\dfrac{D_i}{n_{ijk}}$ and

$$\frac{D_i}{n_{ijk}} - \frac{D_i}{n_{ij(k+1)}} = \left(\frac{1}{n_{ijk}} - \frac{1}{n_{ij(k+1)}}\right) D_i > 0 \quad a.e.,$$

which implies $\mathrm{P}_{\theta_{ij}}\left(\widehat{\theta}_{ij} > x | N_i = k\right)$ in a decreasing function of k.

(M3) Note that

$$\mathrm{P}_{\theta_{ij}}\left(N_i = k | 1 \leq N_i \leq r_i - 1\right) = \binom{r_i}{k} \frac{p_i^k (1-p_i)^{r_i - k}}{1 - p_i^{r_i} - (1-p_i)^{r_i}} \quad \text{for } k = 1,\, 2,\, \ldots,\, r_i - 1.$$

Now for $\theta_{ij}^{'} > \theta_{ij}$, $\dfrac{\mathrm{P}_{\theta_{ij}^{'}}\left(N_i = k | 1 \leq N_i \leq r_i - 1\right)}{\mathrm{P}_{\theta_{ij}}\left(N_i = k | 1 \leq N_i \leq r_i - 1\right)}$ is a increasing function of k and hence N_i is stochastically decreasing in θ_{ij}. □

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