Bayesian Analysis of Simple Step-stress Model under Weibull Lifetimes

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Abstract

Step-stress model is becoming quite popular in recent times for analyzing lifetime data obtained from accelerated life testing experiments. In the usual step-stress experiment, stress levels are allowed to change at each step to get rapid failure of the experimental units. The expected lifetime of the experimental unit is shortened as the stress level increases. The simple step-stress model under different censoring schemes based on Weibull lifetimes is considered in this paper. It is assumed that the lifetime distributions of the experimental units have different scale parameters at different stress levels but they have the same shape parameter. Moreover, the lifetimes satisfy Khamis-Higgins model assumption. It is further assumed that as the stress level increases, the scale parameter also increases. We provide Bayesian inference of the unknown parameters of the Weibull distribution under this order restriction on the scale parameters. Monte Carlo simulations have been performed to see the effectiveness of the proposed method, and a data set has been analyzed for illustrative purposes.

KEY WORDS AND PHRASES: Step-stress life-tests; Khamis-Higgins model; censoring schemes; prior distribution; posterior analysis.

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Acronym and Notation

AE	Average estimate	AL	Average length
ALT	Accelerated life test(ing)	BE	Bayes estimate
CDF	Cumulative distribution function	CEM	Cumulative exposure model
CRI	Credible interval	HPD	Highest posterior density
HCS	Hybrid censoring scheme	KHM	Khamis-Higgins model
MCMC	Markov chain Monte Carlo	MLE	Maximum likelihood estimator(s)
MSE	Mean squares error(s)	PDF	Probability density function
SSLT	Step-stress life test(ing)		

1 Introduction

Long duration of life testing experiment is one of the problems, which is faced by experimenters experimenting with a durable product. Nowadays ALT experiments are gaining popularity to overcome this problem. In an ALT experiment, certain number of units under consideration are put on a life test and are exposed to extreme environmental conditions so that they fail more rapidly than the normal operating condition. It implies that the experimenters observe more failure data within an affordable time. SSLT is a particular type of ALT. A SSLT enables the experimenter to change the stress levels during the experiment. Suppose, n number of items are put on a test at an initial stress level s_1. Let s_2, ..., s_m be $m-1$ stress levels and $\tau_1 < \ldots < \tau_{m-1}$ be $m-1$ prefixed times. At the time point τ_1, the stress level is changed to s_2 from s_1. Similarly, the stress level is changed to s_3 from s_2 at the time τ_2 and so on. Finally, at the time point τ_{k-1}, the stress level is changed from τ_{k-1} to τ_k. The failure times are recorded chronologically. Let $\tau(t) = s_i$ for $\tau_{i-1} \leq t < \tau_i$, $i = 1, 2, \ldots, m$ with $\tau_0 = 0$ and $\tau_m = \infty$. We will call the mapping $\tau(\cdot)$ as step-stress pattern. A simple SSLT is a special case of SSLT, where only two stress levels are considered, and the stress level is changed from s_1 to s_2 at a prefixed time τ_1.

To analyze such data one needs a model that relates the distributions of lifetimes under different stress levels to that of lifetimes under the step-stress pattern. Different models are available in the literature to describe these relationships. Among them, the most popular

one is CEM, first introduced by Seydyakin [18]. Let us assume that the CDF of the lifetime at the stress level s_i is $F_i(\cdot)$ and $F(\cdot)$ is the CDF of lifetime under the step-stress pattern. CEM assumes that the remaining lifetime of an unit depends only on the cumulative exposure accumulated at the current stress level, regardless of how the exposure is actually accumulated, and mathematically it can be written as

$$F(t) = F_i(t - h_{i-1}) \quad \text{for} \quad \tau_{i-1} \leq t < \tau_i, i = 1, 2, \ldots, k,$$

where $\tau_0 = 0$, $\tau_k = \infty$, $h_0 = 0$, and h_i, $i = 2, 3, \ldots, k-1$ is the solution of the equation

$$F_{i+1}(\tau_i - h_i) = F_i(\tau_i - h_{i-1}).$$

This model has been extensively discussed in the literature specially for the exponential lifetimes, see for example Bagdonavicius [1], Nelson [17], and a recent review article by Balakrishnan [4]. Analysis of simple step-stress model has also been performed when lifetimes have a Weibull distribution, mainly under frequentist setup. Analysis of the CEM was performed by Komori [11], when the lifetimes of the experimental units follow Weibull distribution. Inferential aspects of step-stress model under Type-I and Type-II censoring schemes were addressed by Bai and Kim [2] and Kateri and Balakrishnan [8], respectively, when the distribution of lifetimes is assumed to be Weibull. However, it is noticed that MLEs of the unknown parameters of Weibull distribution under SSLT do not exist in closed form and therefore finding MLEs of the unknown parameters involves heavy computations. Most of the further statistical analysis mainly rely on asymptotic distribution of the MLEs. Moreover, the results provided in Bai and Kim [2] and Kateri and Balakrishnan [8] cannot be easily extended to more general censoring situations, *i.e.*, to hybrid and progressive censoring schemes. It seems that Bayesian analysis is a natural choice in this case.

It may be worth mentioning that though some inferential issues on the parameters of Weibull distribution under step-stress model have been addressed in the literature, no attention has been paid to develop the inference under the order restriction on the means of the lifetimes at different stress levels. The frequentist approach to the order restricted inference for parameters of Weibull distribution under step-stress model is quite involved, hence in

this case also Bayesian approach is a natural alternative.

Liu [15] considered a step-stress model for Weibull distributed lifetimes under Bayesian setup when the lifetimes follow CEM. No order restriction on the means of the lifetime at different stress levels was considered in this article. Author used the MCMC technique using Gibbs sampling to obtain BEs and to construct the CRIs of some parametric functions. In this paper we consider a simple SSLT, when the lifetimes are assumed to have two-parameter Weibull distribution with different scale parameters but same shape parameter at two different stress levels. Though CEM is the most popular model in case of exponential lifetimes, it is not mathematically tractable under Weibull lifetimes. Weibull CEM does not transform to exponential CEM under power transformation. An alternative model for Weibull lifetimes is KHM (see Section 3 for more details), proposed by Khamis and Higgins [10]. CDF under KHM assumption coincides with the CDF under CEM assumption for exponentially distributed lifetime under power transform. Moreover, it may be worth mentioning that the KHM and CEM for Weibull distributed lifetimes are difficult to distinguish in practice, see Khamis and Higgins [10]. For these reasons analysis in this article has been performed under KHM assumptions, which is mathematically more tractable than CEM assumptions for Weibull lifetime. We consider order restriction on the means of the lifetime under different stress levels. We use importance sampling technique to obtain BEs and to construct the CRIs of some parametric functions.

Rest of the article is organized as follows. Different censoring schemes and available data are briefly provided in Section 2. Model assumptions and prior information on the unknown parameters are considered in Section 3. In Section 4, we provide the posterior analysis and the Bayes estimators in details for type-I censored data. In Section 5, simulation study has been performed to judge the effectiveness of the procedures described in Section 4, and analysis of a data set has been provided for illustrative purposes. In Section 6, we have indicated how the proposed method can be implemented for other censoring schemes. Finally, we conclude the paper in Section 7.

2 Different Censoring Schemes and Available Data

A total of n units is placed on a simple step-stress life testing experiment. The stress level is changed from s_1 to s_2 at a prefixed time τ_1, and $\tau_2 > \tau_1$ is another prefixed time. The positive integer $r \leq n$ is also pre-fixed. The role of r and τ_2 will be clear later. Let τ^* and n^* denote the termination time of the experiment and total number of failures observed before τ^*, respectively. Note that τ^* and n^* depend on the censoring scheme. Let the ordered lifetimes of the items be denoted by $t_{1:n} < \ldots < t_{n:n}$. Let n_1^* and n_2^* denote the number of failures before the time τ_1 and between τ_1 and τ_2 respectively. They can also be zero. Now we briefly describe different censoring schemes, and available data in each case.

Type-I Censoring Scheme

In a Type-I censoring scheme the experiment is terminated at a prefixed time. For more details of Type-I censoring scheme, readers are referred to Lawless [14], Miller [16], and Bain and Englehardt [3]. A simple SSLT is terminated when the time τ_2 on the test has been reached under Type-I censoring scheme, and the available data are one of the following forms.

(a) $\{\tau_1 < t_{1:n} < \ldots < t_{n_2^*:n} < \tau_2\}$,
(b) $\{t_{1:n} < \ldots < t_{n_1^*:n} < \tau_1 < t_{n_1+1:n} < \ldots < t_{n_1^*+n_2^*:n} < \tau_2\}$,
(c) $\{t_{1:n} < \ldots < t_{n_1^*:n} < \tau_1 < \tau_2\}$.

Type-II Censoring Scheme

In a Type-II censoring scheme, the test is terminated when r-th failure takes place, *i.e.*, it is terminated at a random time $t_{r:n}$. Interested readers are referred to Lawless [14], Miller [16], and Bain and Englehardt [3] for more details of this censoring scheme. The available data from a simple SSLT are one of the following forms under Type-II censoring scheme.

(a) $\{\tau_1 < t_{1:n} < \ldots < t_{r:n}\}$,
(b) $\{t_{1:n} < \ldots < t_{n_1^*:n} < \tau_1 < t_{n_1^*+1:n} < \ldots < t_{r:n}\}$, $n_1 < r$,
(c) $\{t_{1:n} < \ldots < t_{r:n} < \tau_1 < \tau_2\}$.

Type-I Hybrid Censoring Scheme

The test is terminated when r-th failure occurs or time τ_2 is reached on the test, whichever is earlier, *i.e.*, it is terminated at a random time $\tau^* = \min\{t_{r:n}, \tau_2\}$. This censoring scheme was introduced by Epstein [7]. The available data from a simple SSLT are one of the following forms under Type-I HCS.

(a) $\{\tau_1 < t_{1:n} < \ldots < t_{r:n}\}$ if $t_{r:n} < \tau_2$,

(b) $\{t_{1:n} < \ldots < t_{n_1^*:n} < \tau_1 < t_{n_1^*+1:n} < \ldots < t_{r:n}\}$ if $t_{r:n} < \tau_2$, $n_1 < r$,

(c) $\{t_{1:n} < \ldots < t_{r:n} < \tau_1\}$ if $t_{r:n} < \tau_2$,

(d) $\{\tau_1 < t_{1:n} < \ldots < t_{n_2^*:n} < \tau_2\}$ if $t_{r:n} > \tau_2$,

(e) $\{t_{1:n} < \ldots < t_{n_1^*:n} < \tau_1 < t_{n_1^*+1:n} < \ldots < t_{n_1^*+n_2^*:n} < \tau_2\}$ if $t_{r:n} > \tau_2$, $n_1 < r$,

(f) $\{t_{1:n} < \ldots < t_{n_1^*:n} < \tau_1 < \tau_2\}$ if $t_{r:n} > \tau_2$.

Type-II Hybrid Censoring Scheme

This censoring scheme was proposed by Childs et al. [6]. In Type-II HCS, the experiment is terminated when r-th item fails or time τ_2 is reached on the test, whichever is later, *i.e.*, the experiment termination time is $\tau^* = \max\{t_{r:n}, \tau_2\}$. The available data from a simple SSLT are one of the following forms under this censoring scheme.

(a) $\{\tau_1 < t_{1:n} < \ldots < t_{r:n}\}$ if $t_{r:n} \geq \tau_2$,

(b) $\{t_{1:n} < \ldots < t_{n_1^*:n} < \tau_1 < t_{n_1^*+1:n} < \ldots < t_{r:n}\}$ if $t_{r:n} \geq \tau_2$, $n_1 < r$,

(c) $\{\tau_1 < t_{1:n} < \ldots < t_{n_2^*:n} < \tau_2\}$ if $t_{r:n} < \tau_2$,

(d) $\{t_{1:n} < \ldots < t_{n_1^*:n} < \tau_1 < t_{n_1^*+1:n} < \ldots < t_{n_1^*+n_2^*:n} < \tau_2\}$ if $t_{r:n} < \tau_2$,

(e) $\{t_{1:n} < \ldots < t_{n_1^*:n} < \tau_1 < \tau_2\}$ if $t_{r:n} < \tau_2$.

Type-II Progressive Censoring Scheme

Let $R_1, \ldots, R_m$ be m prefixed non-negative integers such that

$$m + \sum_{j=1}^{m} R_j = n.$$

At the time of the first failure, say $t_{1:n}$, R_1 units are chosen at random from the remaining $(n-1)$ units and they are removed from the experiment. Similarly, at the time of the second

failure, say $t_{2:n}$, R_2 units are chosen at random from the remaining $(n - R_1 - 2)$ surviving units and they are removed from the test, and so on. Finally at the time of the mth failure, say $t_{m:n}$, the rest of the $n - m - \sum_{j=1}^{m-1} R_j = R_m$ units are removed and the experiment is stopped. In this case the available data are one of the following forms.

(a) $\{\tau_1 < t_{1:n} < \ldots < t_{m:n}\}$ if $\tau_1 < t_{1:n}$,

(b) $\{t_{1:n} < \ldots < t_{n_1^*:n} < \tau_1 < t_{n_1^*+1:n} < \ldots < t_{m:n}\}$ if $t_{1:n} < \tau_1 < t_{m:n}$,

(c) $\{t_{1:n} < \ldots < t_{m:n} < \tau_1\}$ if $\tau_1 \geq t_{m:n}$.

3 Model Assumption and Prior Information

In this article we consider a simple step-stress life testing, where n units are put on a life testing experiment at the initial stress level s_1. Let τ_1 be a prefixed time at which the stress level is changed from s_1 to s_2. It is assumed that the lifetimes of the experimental units are independently distributed random variables having Weibull distribution. PDF and the CDF of the lifetime under stress level s_i for $i = 1, 2$, are given by

$$f(t;\, \beta,\, \lambda_i) = \beta\lambda_i t^{\beta-1}\, e^{-\lambda_i t^{\beta}} \qquad \text{for} \quad 0 < t < \infty \qquad \beta > 0 \qquad \lambda_i > 0,$$

and

$$F(t;\, \beta,\, \lambda_i) = 1 - e^{-\lambda_i t^{\beta}} \qquad \text{for} \quad 0 < t < \infty \qquad \beta > 0 \qquad \lambda_i > 0, \tag{1}$$

respectively. It is further assumed that the failure time data come from a KHM under the step-stress pattern, hence, it has the following CDF;

$$G(t;\, \beta,\, \lambda_1,\, \lambda_2) = \begin{cases} 1 - e^{-\lambda_1 t^{\beta}} & \text{if } 0 < t < \tau_1 \\ 1 - e^{-\lambda_2(t^{\beta} - \tau_1^{\beta}) - \lambda_1 \tau_1^{\beta}} & \text{if } \tau_1 \leq t < \infty. \end{cases}$$

The corresponding PDF is given by

$$g(t;\, \beta,\, \lambda_1,\, \lambda_2) = \begin{cases} \beta\lambda_1\, t^{\beta-1}\, e^{-\lambda_1 t^{\beta}} & \text{if } 0 < t < \tau_1 \\ \beta\lambda_2\, t^{\beta-1}\, e^{-\lambda_2(t^{\beta} - \tau_1^{\beta}) - \lambda_1 \tau_1^{\beta}} & \text{if } \tau_1 \leq t < \infty. \end{cases}$$

For developing the Bayesian inference, we need to assume some priors on the unknown

parameters. Suppose λ_1 and λ_2 are independently distributed according to gamma distribution. If β is known, they are conjugate priors for λ_1 and λ_2. However, following the argument of Soland [19] it can be shown that there does not exist any continuous conjugate prior for (β, λ_1, λ_2). A continuous-discrete conjugate prior do exist, where continuous part corresponds to the scale parameters and discrete part corresponds to the shape parameter. Khaminskiy and Krivtsov [9] criticized this choice of priors as it is difficulty to apply in real life, hence it is not addressed further.

Following the approach of Berger and Sun [5], Kundu and Gupta [13], and Kundu [12], here we assume that λ_i has a gamma prior with shape and scale parameters $a_i > 0$ and $b_i > 0$, respectively, *i.e.*, the prior assumption on λ_i is summarized in the following PDF.

$$\pi_i(\lambda_i) \propto \lambda_i^{a_i-1}\, e^{-\lambda_i b_i}\,; \quad \lambda_i > 0, \quad i = 1,\, 2. \tag{2}$$

The prior on the shape parameter β is also assumed to be a gamma distribution with shape and scale parameter $a_3 > 0$ and $b_3 > 0$, respectively, *i.e.*, the prior PDF of β is given by

$$\pi_3(\beta) \propto \beta^{a_3-1}\, e^{-b_3\beta} \quad \text{for} \quad \beta > 0. \tag{3}$$

It is further assumed that β, λ_1, and λ_2 are independently distributed. We discuss the posterior analysis of type-I censored data in details in Section 4.1 under this prior assumptions.

Next we consider order restricted inference of the parameters under the same model assumptions. Note that the main aim of a SSLT is to get rapid failures by imposing severe stress level on the products under test. Hence, it is natural to assume that the mean lifetime at the stress level s_1 is greater than the mean lifetime at the stress level s_2, which implies $\lambda_1 < \lambda_2$ under lifetime distribution (1). Therefore, one of the ways to incorporate order restriction is to assume that $\lambda_1 = \alpha\lambda_2$ with $0 < \alpha < 1$. The following priors are assumed under this order restricted situation. It is assumed that priors on β and λ_2 are same as the previous case, *i.e.*, they have priors $\pi_2(\cdot)$ and $\pi_3(\cdot)$, respectively, and α has a beta prior, with parameters $a_4 > 0$ and $b_4 > 0$, having PDF

$$\pi_4(\alpha) \propto \alpha^{a_4-1}\,(1-\alpha)^{b_4-1} \quad \text{for} \quad 0 < \alpha < 1. \tag{4}$$

Here also we assume that α, β, and λ_2 are independently distributed. Therefore, the joint prior PDF of (λ_1, λ_2) can be written as;

$$\pi(\lambda_1, \lambda_2) = \frac{b^a}{\Gamma(a)B(c,d)} \lambda_2^{a-c-d} e^{-b\lambda_2} \lambda_1^{c-1} (\lambda_2 - \lambda_1)^{d-1} \quad \text{for} \quad 0 < \lambda_1 < \lambda_2 < \infty.$$

As the joint prior on (λ_1, λ_2) is complicated, a gray-scale plot is provided in Figure 1 for different values of hyper-parameters. In the plot black color represents the maximum value of the density function, whereas white color represents the minimum value, which is zero in all the plots. We have taken $b = 1.0$ only, as different values of b only affects the spread of the density function for fixed shape parameter.

4 Posterior Analysis under Type-I Censoring Scheme

4.1 Under Unrestricted Prior Assumption

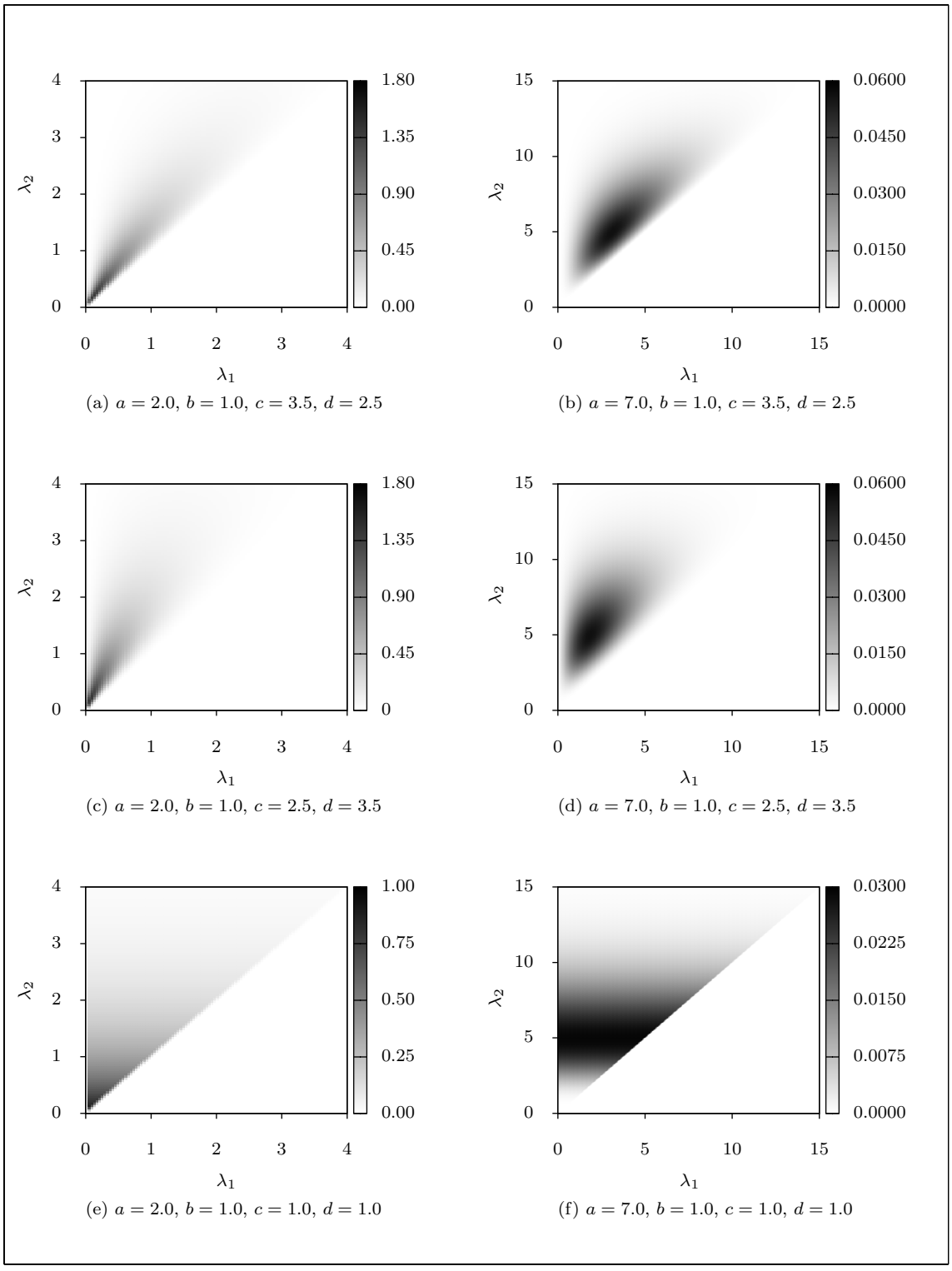
In case of Type-I censoring scheme, $\tau^* = \tau_2$. For Case (a): $n_1^* = 0$, $n_2^* = n_2 \leq n$, Case (b): $n_1^* = n_1 > 0$, $n_2^* = n_2 > 0$, Case (c): $n_1^* = n_1 > 0$, $n_2^* = 0$. Let $n^* = n_1^* + n_2^*$. Based on the observations from a simple SSLT under Type-I censoring scheme, the likelihood function can be written as

$$l_1(\text{Data}\,|\,\beta,\, \lambda_1,\, \lambda_2) \propto \beta^{n_1^*+n_2^*}\, \lambda_1^{n_1^*}\, \lambda_2^{n_2^*} \left(\prod_{i=1}^{n_1^*+n_2^*} t_{i:n} \right)^{\beta-1} e^{-\lambda_1 D_1(\beta) - \lambda_2 D_2(\beta)}, \tag{5}$$

where $D_1(\beta) = \sum_{j=1}^{n_1^*} t_{j:n}^{\beta} + (n - n_1^*)\tau_1^{\beta}$ and $D_2(\beta) = \sum_{j=n_1^*+1}^{n^*} (t_{j:n}^{\beta} - \tau_1^{\beta}) + (n - n^*)(\tau^{*\beta} - \tau_1^{\beta})$. Therefore, based on the priors $\pi_1(\cdot)$, $\pi_2(\cdot)$, and $\pi_3(\cdot)$ mentioned above posterior PDF of β, λ_1, and λ_2 becomes

$$l_2(\beta,\, \lambda_1,\, \lambda_2\,|\,\text{Data}) \propto \beta^{n^*+a_3-1}\, \lambda_1^{n_1^*+a_1-1}\, \lambda_2^{n_2+a_2-1}\, e^{-(b_3-c_1)\,\beta - \lambda_1\, A_1(\beta) - \lambda_2\, A_2(\beta)}$$
$$\text{if} \quad \beta > 0,\, \lambda_1 > 0,\, \lambda_2 > 0, \tag{6}$$

where $A_1(\beta) = b_1 + D_1(\beta)$, $A_2(\beta) = b_2 + D_2(\beta)$, and $c_1 = \sum_{i=1}^{n^*} \ln t_{i:n}$. Note that the right hand side of (6) is integrable if we take proper priors on the unknown parameters, see Appendix A.1 for details. If we want to compute the BEs of some functions of β, λ_1,



(a) $a = 2.0$, $b = 1.0$, $c = 3.5$, $d = 2.5$

(b) $a = 7.0$, $b = 1.0$, $c = 3.5$, $d = 2.5$

(c) $a = 2.0$, $b = 1.0$, $c = 2.5$, $d = 3.5$

(d) $a = 7.0$, $b = 1.0$, $c = 2.5$, $d = 3.5$

(e) $a = 2.0$, $b = 1.0$, $c = 1.0$, $d = 1.0$

(f) $a = 7.0$, $b = 1.0$, $c = 1.0$, $d = 1.0$

Figure 1: Plot of prior density for different values of hyper-parameters.

and λ_2, say $g(\beta,\, \lambda_1,\, \lambda_2)$, with respect to the squared error loss function, it will be posterior expectation of $g(\beta,\, \lambda_1,\, \lambda_2)$, *i.e.*,

$$\widehat{g}(\beta,\, \lambda_1,\, \lambda_2) = \int_0^\infty \int_0^\infty \int_0^\infty g(\beta,\, \lambda_1,\, \lambda_2) l_2(\beta,\, \lambda_1,\, \lambda_2 \,|\, \text{Data})\, d\lambda_2 d\lambda_1 d\beta. \tag{7}$$

Unfortunately, the closed form of (7) cannot be obtained in most of the cases. One may use numerical techniques to compute (7). Alternatively, other approximation can be used to compute (7). However, CRI for a parametric function cannot be constructed by these numerical methods. Hence, we propose to use an importance sampling technique to compute Bayes estimate as well as to construct CRI of a parametric function. Note that

$$l_2(\beta,\, \lambda_1,\, \lambda_2 \,|\, \text{Data}) = l_3(\lambda_1 \,|\, \beta,\, \text{Data}) \times l_4(\lambda_2 \,|\, \beta,\, \text{Data}) \times l_5(\beta \,|\, \text{Data}),$$

where

$$l_3(\lambda_1 \,|\, \beta,\, \text{Data}) = \frac{\{A_1(\beta)\}^{n_1^*+a_1}}{\Gamma(n_1^*+a_1)}\, \lambda_1^{n_1^*+a_1-1}\, e^{-\lambda_1 A_1(\beta)} \quad \text{if} \quad \lambda_1 > 0, \tag{8}$$

$$l_4(\lambda_2 \,|\, \beta,\, \text{Data}) = \frac{\{A_2(\beta)\}^{n_2^*+a_2}}{\Gamma(n_2^*+a_2)}\, \lambda_2^{n_2^*+a_2-1}\, e^{-\lambda_2 A_2(\beta)} \quad \text{if} \quad \lambda_2 > 0, \tag{9}$$

and

$$l_5(\beta \,|\, \text{Data}) = c_2 \frac{\beta^{n^*+a_3-1} e^{-(b_3-c_1)\beta}}{\{A_1(\beta)\}^{n_1^*+a_1} \{A_2(\beta)\}^{n_2^*+a_2}} \quad \text{if} \quad \beta > 0. \tag{10}$$

The normalizing constant c_2 in (10) can be found using numerical method. Though it is not easy to prove the log-concavity of the $l_5(\beta \,|\, \text{Data})$, the plots (see Figure 2) suggest that $l_5(\beta \,|\, \text{Data})$ is a unimodal function. Hence, we try to approximate $l_5(\beta \,|\, \text{Data})$ by a gamma density function using similar idea as in Kundu [12], where the parameters of the gamma distribution are determined by equating mean and variance of $l_5(\beta \,|\, \text{Data})$ to those of a gamma distribution. Let m_1 and m_2 denote the mean and variance, respectively, corresponding to the density $l_5(\beta \,|\, \text{Data})$. The shape and scale parameters of the approximating gamma distribution are given by $a_5 = \dfrac{m_1^2}{m_2}$ and $b_5 = \dfrac{m_1}{m_2}$, respectively. Let us define

$$l_6(\beta \,|\, \text{Data}) = \frac{b_5^{a_5}}{\Gamma(a_5)}\, \beta^{a_5-1}\, e^{-b_5\beta} \quad \text{for} \quad \beta > 0.$$

Note that $l_2(\beta,\, \lambda_1,\, \lambda_2 \,|\, \text{Data})$ can be expressed as follows.

$$l_2(\beta,\, \lambda_1,\, \lambda_2 \,|\, \text{Data}) = w_1(\beta) \times l_3(\lambda_1 \,|\, \beta,\, \text{Data}) \times l_4(\lambda_2 \,|\, \beta,\, \text{Data}) \times l_6(\beta \,|\, \text{Data}),$$

where $w_1(\beta) = \dfrac{l_5(\beta \,|\, \text{Data})}{l_6(\beta \,|\, \text{Data})}$. Now we propose to use the following algorithm based on importance sampling technique to compute BE and to construct the CRI of function $g(\beta,\, \lambda_1,\, \lambda_2)$.

Algorithm 4.1

Step 1. Generate β_1 from Gamma$(a_5,\, b_5)$ distribution.

Step 2. For the given β_1, generate λ_{11} from (8).

Step 3. For the given β_1, generate λ_{21} from (9).

Step 4. Continue the process M times to get $\{(\beta_1,\, \lambda_{11},\, \lambda_{21}),\, \ldots,\, (\beta_M,\, \lambda_{1M},\, \lambda_{2M})\}$.

Step 5. Compute $g_i = g(\beta_i,\, \lambda_{1i},\, \lambda_{2i});\ i = 1,\, 2,\, \ldots,\, M$.

Step 6. Calculate the weights $w_{1i} = w_1(\beta_i);\ i = 1,\, 2,\, \ldots,\, M$.

Step 7. Compute the BE of $g(\beta,\, \lambda_1,\, \lambda_2)$ as

$$\widehat{g}_{BE}(\beta,\, \lambda_1,\, \lambda_2) = \frac{1}{M} \sum_{j=1}^{M} w_{1j} g_j.$$

Step 8. To construct a $100(1-\gamma)\%$ CRI of $g(\beta,\, \lambda_1,\, \lambda_2)$, first order g_j for $j = 1,\, \ldots,\, M$, say $g_{(1)} < g_{(2)} < \ldots < g_{(M)}$, and order w_j accordingly to get $w_{1(1)},\, w_{1(2)},\, \ldots,\, w_{1(M)}$. Note that $w_{1(1)},\, w_{1(2)},\, \ldots,\, w_{1(M)}$ may not be ordered. A $100(1-\gamma)\%$ CRI can be obtained as $(g_{(j_1)}, g_{(j_2)})$, where j_1 and j_2 satisfy

$$j_1,\, j_2 \in \{1,\, 2,\, \ldots,\, M\}, \qquad j_1 < j_2, \qquad \frac{1}{M} \sum_{i=j_1}^{j_2} w_{1(i)} \le 1-\gamma < \frac{1}{M} \sum_{i=j_1}^{j_2+1} w_{1(i)}. \tag{11}$$

The $100(1-\gamma)\%$ HPD CRI of $g(\beta,\, \lambda_1,\, \lambda_2)$ becomes $(g_{(j_1^*)},\, g_{(j_2^*)})$, where $j_1^* < j_2^*$, $j_1^*,\, j_2^* \in \{1,\, 2,\, \ldots,\, M\}$ satisfy

$$\frac{1}{M} \sum_{i=j_1^*}^{j_2^*} w_{1(i)} \le 1-\gamma < \frac{1}{M} \sum_{i=j_1^*}^{j_2^*+1} w_{1(i)}, \qquad\qquad g_{(j_2^*)} - g_{(j_1^*)} \le g_{(j_2)} - g_{(j_1)},$$

for all j_1 and j_2 satisfying (11).

4.2 Under Order Restricted Prior Assumption

Computations of BE and construction of associated CRI of some parametric function $g(\beta,\, \lambda_1,\, \lambda_2)$ under order restricted priors are addressed in this subsection. Using the reparameterization

$\lambda_1 = \alpha\,\lambda_2$ $(0 < \alpha < 1)$ and (2), (3), (4), and (5), one can express the posterior density function of $(\alpha,\, \beta,\, \lambda_2)$ as

$$l_7(\alpha,\, \beta,\, \lambda_2 \,|\, \text{Data}) \propto \alpha^{n_1^*+a_4-1}(1-\alpha)^{b_4-1}\beta^{n^*+a_3-1}\lambda_2^{n^*+a_2-1} \times e^{-\lambda_2(\alpha D_1(\beta)+D_2(\beta)+b_2)-(b_3-c_1)\beta} \quad \text{if } 0<\alpha<1,\, \beta>0,\, \lambda_2>0. \tag{12}$$

Like in the previous case, the right hand side of (12) is integrable if proper priors are assumed on the unknown parameters, see Appendix A.2 for details. Now under squared error loss function BE of parametric function $g(\alpha,\, \beta,\, \lambda_2)$ is given by

$$\widehat{g}(\alpha,\, \beta,\, \lambda_2) = \int_0^1\int_0^\infty\int_0^\infty g(\alpha,\, \beta,\, \lambda_2) l_7(\alpha,\, \beta,\, \lambda_2 \,|\, \text{Data})\, d\lambda_2 d\beta d\alpha.$$

Note that

$$l_7(\alpha,\, \beta,\, \lambda_2 \,|\, \text{Data}) \propto w_2(\alpha,\, \beta) \times l_8(\lambda_2 \,|\, \alpha,\, \beta,\, \text{Data}) \times l_9(\beta \,|\, \text{Data}),$$

where

$$w_2(\alpha,\, \beta) = \frac{\alpha^{n_1^*+a_4-1}(1-\alpha)^{b_4-1}\, e^{c_2\beta}}{\{\alpha D_1(\beta) + D_2(\beta) + b_2\}^{n^*+a_2}},$$

$$l_8(\lambda_2 \,|\, \alpha,\, \beta,\, \text{Data}) = \frac{\{\alpha D_1(\beta) + D_2(\beta) + b_2\}^{n^*+a_2}}{\Gamma(n^*+a_2)}\lambda_2^{n^*+a_2-1}\, e^{-\lambda_2(\alpha D_1(\beta)+D_2(\beta)+b_2)}, \tag{13}$$

and

$$l_9(\beta \,|\, \text{Data}) = \frac{(b_3-c_1)^{n^*+a_3}}{\Gamma(n^*+a_3)}\beta^{n^*+a_3-1}\, e^{-(b_3-c_1+c_2)\beta}, \tag{14}$$

with $c_2 = n_3^*\tau_2^*$, n_3^* is the number of failure times which are less than one, and

$$\tau_2^* = \begin{cases} \tau_2 & \text{if } n-n^*>0 \\ t_{n:n} & \text{if } n-n^*=0. \end{cases}$$

Depending upon the previous expression of $l_7(\alpha,\, \beta,\, \lambda_2 \,|\, \text{Data})$, the following algorithm is proposed to compute BE as well as to construct CRI.

Algorithm 4.2

Step 1. Generate α_1 from U(0, 1) distribution.

Step 2. Generate β_1 from (14).

Step 3. For the given α_1 and β_1, generate λ_{21} from (13).

Step 4. Continue the process M times to get $\{(\alpha_1, \beta_1, \lambda_{21}), \ldots, (\alpha_M, \beta_M, \lambda_{2M})\}$.

Step 5. Calculate $g_i = g(\alpha_i, \beta_i, \lambda_{2i});\ i = 1, 2, \ldots, M$.

Step 6. Calculate the weights $w_{2i} = w_2(\alpha_i, \beta_i);\ i = 1, 2, \ldots, M$.

Step 7. Calculate the normalize weights $w^*_{2i} = \dfrac{w_{2i}}{\sum_{j=1}^{M} w_{2j}};\ i = 1, 2, \ldots, M$.

Step 8. Compute the BE of $g(\alpha, \beta, \lambda_2)$ as $\widehat{g}_{BE}(\beta, \lambda_1, \lambda_2) = \sum_{j=1}^{M} w^*_{2j} g_j$.

Step 9. To construct a $100(1-\gamma)\%$ CRI of $g(\alpha, \beta, \lambda_2)$, first order g_j for $j = 1, \ldots, M$, say $g_{(1)} < g_{(2)} < \ldots < g_{(M)}$, and order w^*_{2j} accordingly to get $w^*_{2(1)}, w^*_{2(2)}, \ldots, w^*_{2(M)}$. Note that $w^*_{2(1)}, w^*_{2(2)}, \ldots, w^*_{2(M)}$ may not be ordered. A $100(1-\gamma)\%$ CRI can be obtained as $(g_{(j_1)}, g_{(j_2)})$, where j_1 and j_2 satisfy

$$j_1, j_2 \in \{1, 2, \ldots, M\}, \qquad j_1 < j_2, \qquad \sum_{i=j_1}^{j_2} w^*_{2(i)} \leq 1-\gamma < \sum_{i=j_1}^{j_2+1} w^*_{2(i)}. \tag{15}$$

The $100(1-\gamma)\%$ HPD CRI of $g(\alpha, \beta, \lambda_2)$ becomes $(g_{(j_1^*)}, g_{(j_2^*)})$, where $j_1^* < j_2^*$, $j_1^*, j_2^* \in \{1, 2, \ldots, M\}$ satisfy

$$\sum_{i=j_1^*}^{j_2^*} w^*_{2(i)} \leq 1-\gamma < \sum_{i=j_1^*}^{j_2^*+1} w^*_{2(i)}, \qquad g_{(j_2^*)} - g_{(j_1^*)} \leq g_{(j_2)} - g_{(j_1)},$$

for all j_1 and j_2 satisfying (15).

5 Simulations and Data Analysis

5.1 Simulation Results

In this section we present some simulation results to judge the performance of the proposed procedures in the Sections 4.1 and 4.2 for different values of τ_1, τ_2 and n. Here we choose two sets of priors. Prior I: $a_1 = 0.0001$, $b_1 = 0.0001$, $a_2 = 0.0001$, $b_2 = 0.0001$, $a_3 = 0.0001$, $b_3 = 0.0001$, $a_4 = 1$, and $b_4 = 1$. Prior II : $a_1 = 64.0$, $b_1 = 80.0$, $a_2 = 48.5$, $b_2 = 22.0$, $a_3 = 40.0$, $b_3 = 20.0$, $a_4 = 4.41$, and $b_4 = 7.7$. Note that the priors on β, λ_1 and λ_2 are assumed to be very flat in unrestricted case for Prior I, and they are 'almost' non-informative.

Table 1: AEs and MSEs of BEs of β, λ_1, and λ_2 for unrestricted case.

			Prior I						Prior II					
			β		λ_1		λ_2		β		λ_1		λ_2	
n	τ_1	τ_2	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
40	0.60	0.80	2.180	0.5033	1.087	0.4869	2.477	0.5826	2.003	0.0674	0.799	0.0088	2.205	0.0738
		1.20	2.105	0.3179	0.981	0.2372	2.429	0.4292	1.999	0.0557	0.801	0.0088	2.204	0.0680
	0.65	0.85	2.159	0.4464	0.997	0.2378	2.438	0.5955	2.001	0.0665	0.799	0.0088	2.204	0.0795
		1.25	2.099	0.2961	0.942	0.1559	2.437	0.4792	2.006	0.0549	0.799	0.0086	2.206	0.0706
	0.70	0.90	2.139	0.3707	0.939	0.1341	2.415	0.6383	2.003	0.0636	0.800	0.0085	2.204	0.0777
		1.30	2.103	0.2742	0.918	0.1143	2.424	0.5191	1.999	0.0548	0.800	0.0089	2.204	0.0732
	0.75	0.95	2.118	0.3147	0.902	0.0922	2.426	0.6838	1.997	0.0620	0.800	0.0080	2.205	0.0818
		1.35	2.105	0.2601	0.899	0.0848	2.425	0.5768	1.997	0.0523	0.801	0.0083	2.203	0.0756
	0.80	1.00	2.110	0.2934	0.881	0.0669	2.406	0.7211	2.002	0.0589	0.799	0.0081	2.207	0.0795
		1.40	2.111	0.2478	0.886	0.0650	2.435	0.6153	2.001	0.0529	0.800	0.0082	2.205	0.0725
50	0.60	0.80	2.153	0.4111	1.028	0.3297	2.405	0.4110	2.002	0.0622	0.800	0.0088	2.205	0.0750
		1.20	2.071	0.2375	0.943	0.1611	2.395	0.3357	1.999	0.0507	0.801	0.0089	2.210	0.0620
	0.65	0.85	2.136	0.3322	0.959	0.1586	2.374	0.4159	2.002	0.0607	0.800	0.0087	2.206	0.0748
		1.25	2.085	0.2258	0.927	0.1198	2.384	0.3453	1.998	0.0521	0.800	0.0086	2.206	0.0666
	0.70	0.90	2.112	0.2841	0.921	0.1014	2.354	0.4610	2.003	0.0601	0.799	0.0080	2.204	0.0764
		1.30	2.082	0.2115	0.905	0.0843	2.384	0.4023	2.000	0.0508	0.800	0.0080	2.201	0.0673
	0.75	0.95	2.101	0.2445	0.895	0.0706	2.354	0.4933	2.000	0.0564	0.801	0.0081	2.206	0.0789
		1.35	2.091	0.2005	0.889	0.0638	2.369	0.4214	1.998	0.0485	0.801	0.0080	2.202	0.0696
	0.80	1.00	2.084	0.2096	0.874	0.0525	2.367	0.5525	2.001	0.0541	0.800	0.0077	2.208	0.0783
		1.40	2.091	0.1890	0.876	0.0495	2.358	0.4355	2.002	0.0469	0.800	0.0078	2.207	0.0723

Table 2: ALs of Symmetric CRI of β for unrestricted case.

			Prior I			Prior II		
n	τ_1	τ_2	90%	95%	99%	90%	95%	99%
40	0.60	0.80	2.124	2.534	3.340	0.836	0.997	1.311
		1.20	1.753	2.084	2.721	0.782	0.932	1.225
	0.65	0.85	1.966	2.346	3.095	0.829	0.989	1.301
		1.25	1.694	2.014	2.634	0.777	0.926	1.217
	0.70	0.90	1.828	2.181	2.876	0.821	0.978	1.287
		1.30	1.638	1.949	2.554	0.764	0.911	1.198
	0.75	0.95	1.707	2.037	2.686	0.807	0.962	1.267
		1.35	1.578	1.878	2.464	0.752	0.897	1.180
	0.80	1.00	1.608	1.919	2.530	0.797	0.950	1.250
		1.40	1.522	1.813	2.380	0.743	0.886	1.165
50	0.60	0.80	1.877	2.239	2.950	0.804	0.958	1.259
		1.20	1.552	1.845	2.413	0.747	0.890	1.170
	0.65	0.85	1.738	2.073	2.732	0.797	0.950	1.249
		1.25	1.506	1.792	2.346	0.739	0.881	1.158
	0.70	0.90	1.609	1.920	2.530	0.788	0.939	1.235
		1.30	1.450	1.726	2.261	0.730	0.870	1.144
	0.75	0.95	1.508	1.799	2.371	0.774	0.922	1.214
		1.35	1.402	1.670	2.191	0.718	0.856	1.126
	0.80	1.00	1.418	1.691	2.227	0.761	0.907	1.194
		1.40	1.349	1.607	2.111	0.708	0.844	1.110

Table 3: ALs of HPD CRI of β for unrestricted case.

			Prior I			Prior II		
n	τ_1	τ_2	90%	95%	99%	90%	95%	99%
40	0.60	0.80	2.077	2.477	3.257	0.831	0.990	1.298
		1.20	1.734	2.059	2.678	0.777	0.926	1.213
	0.65	0.85	1.927	2.298	3.025	0.824	0.981	1.287
		1.25	1.673	1.989	2.593	0.772	0.919	1.206
	0.70	0.90	1.795	2.140	2.816	0.815	0.971	1.273
		1.30	1.618	1.925	2.515	0.759	0.904	1.186
	0.75	0.95	1.679	2.003	2.635	0.802	0.955	1.254
		1.35	1.559	1.855	2.427	0.748	0.891	1.168
	0.80	1.00	1.584	1.890	2.484	0.791	0.943	1.238
		1.40	1.504	1.791	2.345	0.738	0.880	1.154
50	0.60	0.80	1.843	2.197	2.888	0.799	0.951	1.247
		1.20	1.536	1.826	2.380	0.743	0.885	1.160
	0.65	0.85	1.709	2.038	2.679	0.792	0.943	1.237
		1.25	1.491	1.773	2.314	0.735	0.875	1.148
	0.70	0.90	1.585	1.890	2.485	0.783	0.932	1.223
		1.30	1.435	1.707	2.231	0.726	0.865	1.134
	0.75	0.95	1.488	1.774	2.332	0.769	0.916	1.202
		1.35	1.388	1.652	2.162	0.714	0.851	1.115
	0.80	1.00	1.400	1.669	2.193	0.756	0.901	1.182
		1.40	1.336	1.590	2.083	0.704	0.838	1.099

Table 4: ALs of Symmetric CRI of λ_1 for unrestricted case.

			Prior I			Prior II		
n	τ_1	τ_2	90%	95%	99%	90%	95%	99%
40	0.60	0.80	1.803	2.322	3.712	0.309	0.368	0.485
		1.20	1.392	1.723	2.482	0.309	0.368	0.485
	0.65	0.85	1.384	1.734	2.584	0.306	0.365	0.480
		1.25	1.183	1.451	2.043	0.305	0.364	0.479
	0.70	0.90	1.115	1.370	1.945	0.303	0.361	0.475
		1.30	1.028	1.251	1.730	0.302	0.360	0.474
	0.75	0.95	0.937	1.138	1.569	0.299	0.357	0.469
		1.35	0.905	1.094	1.490	0.299	0.357	0.469
	0.80	1.00	0.820	0.989	1.339	0.296	0.352	0.464
		1.40	0.809	0.974	1.312	0.296	0.353	0.464
50	0.60	0.80	1.497	1.889	2.867	0.305	0.364	0.479
		1.20	1.187	1.456	2.057	0.305	0.364	0.479
	0.65	0.85	1.171	1.447	2.083	0.302	0.360	0.473
		1.25	1.035	1.262	1.751	0.301	0.359	0.473
	0.70	0.90	0.965	1.177	1.639	0.298	0.355	0.467
		1.30	0.901	1.092	1.493	0.297	0.354	0.466
	0.75	0.95	0.826	0.999	1.362	0.294	0.350	0.461
		1.35	0.799	0.963	1.302	0.293	0.350	0.460
	0.80	1.00	0.723	0.870	1.170	0.290	0.345	0.454
		1.40	0.715	0.859	1.151	0.289	0.345	0.454

Table 5: ALs of HPD CRI of λ_1 for unrestricted case.

			Prior I			Prior II		
n	τ_1	τ_2	90%	95%	99%	90%	95%	99%
40	0.60	0.80	1.551	2.000	3.204	0.307	0.366	0.480
		1.20	1.270	1.575	2.275	0.307	0.366	0.480
	0.65	0.85	1.249	1.564	2.329	0.304	0.362	0.475
		1.25	1.040	1.278	1.811	0.303	0.362	0.475
	0.70	0.90	0.974	1.185	1.637	0.301	0.358	0.470
		1.30	1.102	1.353	1.906	0.300	0.358	0.470
	0.75	0.95	0.894	1.085	1.493	0.297	0.354	0.465
		1.35	0.868	1.050	1.428	0.297	0.354	0.465
	0.80	1.00	0.793	0.956	1.291	0.294	0.350	0.459
		1.40	0.784	0.944	1.268	0.294	0.350	0.460
50	0.60	0.80	1.330	1.681	2.551	0.303	0.362	0.475
		1.20	1.103	1.355	1.915	0.303	0.361	0.474
	0.65	0.85	1.082	1.337	1.923	0.300	0.357	0.469
		1.25	0.978	1.192	1.654	0.299	0.356	0.468
	0.70	0.90	0.914	1.115	1.549	0.296	0.352	0.463
		1.30	0.863	1.046	1.428	0.295	0.352	0.462
	0.75	0.95	0.796	0.962	1.308	0.292	0.348	0.457
		1.35	0.773	0.931	1.257	0.291	0.347	0.456
	0.80	1.00	0.704	0.846	1.135	0.288	0.343	0.450
		1.40	0.698	0.838	1.120	0.287	0.343	0.450

Table 6: ALs of Symmetric CRI of λ_2 for unrestricted case.

			Prior I			Prior II		
n	τ_1	τ_2	90%	95%	99%	90%	95%	99%
40	0.60	0.80	2.393	2.919	4.097	0.919	1.096	1.442
		1.20	2.057	2.529	3.616	0.853	1.018	1.340
	0.65	0.85	2.402	2.927	4.085	0.921	1.098	1.446
		1.25	2.171	2.660	3.764	0.864	1.031	1.357
	0.70	0.90	2.455	2.989	4.158	0.924	1.102	1.451
		1.30	2.264	2.769	3.890	0.875	1.044	1.375
	0.75	0.95	2.555	3.111	4.320	0.929	1.108	1.458
		1.35	2.360	2.882	4.027	0.886	1.057	1.392
	0.80	1.00	2.628	3.198	4.428	0.934	1.114	1.466
		1.40	2.456	2.994	4.166	0.897	1.069	1.408
50	0.60	0.80	2.060	2.496	3.446	0.896	1.068	1.405
		1.20	1.798	2.193	3.068	0.825	0.984	1.295
	0.65	0.85	2.075	2.511	3.444	0.899	1.071	1.410
		1.25	1.880	2.288	3.180	0.837	0.998	1.314
	0.70	0.90	2.138	2.590	3.552	0.902	1.076	1.416
		1.30	1.977	2.403	3.324	0.848	1.012	1.333
	0.75	0.95	2.217	2.686	3.681	0.908	1.083	1.426
		1.35	2.050	2.488	3.429	0.861	1.027	1.353
	0.80	1.00	2.308	2.797	3.829	0.914	1.091	1.436
		1.40	2.122	2.575	3.538	0.875	1.043	1.374

Table 7: ALs of HPD CRI of λ_2 for unrestricted case.

			Prior I			Prior II		
n	τ_1	τ_2	90%	95%	99%	90%	95%	99%
40	0.60	0.80	2.296	2.792	3.883	0.913	1.088	1.428
		1.20	1.957	2.399	3.399	0.847	1.010	1.327
	0.65	0.85	2.302	2.798	3.877	0.914	1.090	1.431
		1.25	2.065	2.526	3.551	0.858	1.023	1.343
	0.70	0.90	2.348	2.854	3.948	0.918	1.093	1.436
		1.30	2.152	2.628	3.676	0.869	1.036	1.361
	0.75	0.95	2.436	2.963	4.100	0.922	1.099	1.443
		1.35	2.243	2.736	3.810	0.880	1.049	1.377
	0.80	1.00	2.500	3.040	4.201	0.927	1.105	1.451
		1.40	2.333	2.842	3.945	0.890	1.061	1.393
50	0.60	0.80	1.996	2.413	3.303	0.890	1.060	1.391
		1.20	1.728	2.105	2.924	0.819	0.977	1.282
	0.65	0.85	2.008	2.428	3.311	0.892	1.063	1.396
		1.25	1.807	2.197	3.036	0.831	0.990	1.301
	0.70	0.90	2.063	2.496	3.409	0.896	1.068	1.402
		1.30	1.898	2.306	3.177	0.842	1.004	1.319
	0.75	0.95	2.134	2.584	3.529	0.902	1.075	1.411
		1.35	1.967	2.387	3.279	0.855	1.019	1.339
	0.80	1.00	2.217	2.685	3.668	0.908	1.082	1.421
		1.40	2.036	2.469	3.385	0.868	1.035	1.360

Table 8: AEs and MSEs of BEs of β, λ_1, and λ_2 for order restricted case.

			Prior I						Prior II					
			β		λ_1		λ_2		β		λ_1		λ_2	
n	τ_1	τ_2	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
40	0.60	0.80	2.156	0.2940	0.984	0.1186	2.302	0.3892	2.011	0.0727	0.800	0.0443	2.202	0.0749
		1.20	2.201	0.2457	0.997	0.1057	2.250	0.2503	1.969	0.0610	0.792	0.0425	2.202	0.0649
	0.65	0.85	2.191	0.3094	0.980	0.1040	2.272	0.3565	2.004	0.0707	0.807	0.0396	2.205	0.0723
		1.25	2.212	0.2410	0.976	0.0872	2.219	0.2576	1.977	0.0608	0.802	0.0376	2.210	0.0643
	0.70	0.90	2.215	0.3279	0.962	0.0866	2.233	0.3652	2.011	0.0710	0.800	0.0353	2.201	0.0739
		1.30	2.219	0.2404	0.960	0.0727	2.185	0.2832	1.977	0.0585	0.802	0.0346	2.210	0.0672
	0.75	0.95	2.196	0.2990	0.938	0.0668	2.226	0.4216	2.004	0.0623	0.807	0.0338	2.204	0.0727
		1.35	2.242	0.2616	0.944	0.0604	2.168	0.3476	1.975	0.0561	0.798	0.0334	2.205	0.0668
	0.80	1.00	2.197	0.2703	0.923	0.0565	2.186	0.4282	2.000	0.0621	0.801	0.0304	2.202	0.0727
		1.40	2.264	0.3077	0.932	0.0507	2.134	0.3644	1.979	0.0556	0.796	0.0299	2.204	0.0690
50	0.60	0.80	2.110	0.2156	0.958	0.0915	2.250	0.2940	2.013	0.0686	0.806	0.0385	2.202	0.0692
		1.20	2.185	0.1990	0.985	0.0947	2.226	0.1973	1.961	0.0604	0.787	0.0369	2.204	0.0565
	0.65	0.85	2.158	0.2430	0.961	0.0851	2.239	0.3071	2.008	0.0681	0.806	0.0340	2.205	0.0694
		1.25	2.200	0.2048	0.965	0.0750	2.198	0.2100	1.967	0.0571	0.797	0.0331	2.209	0.0607
	0.70	0.90	2.169	0.2376	0.943	0.0676	2.214	0.3092	2.007	0.0644	0.807	0.0314	2.205	0.0704
		1.30	2.206	0.2089	0.948	0.0601	2.167	0.2413	1.968	0.0550	0.788	0.0305	2.209	0.0611
	0.75	0.95	2.168	0.2304	0.922	0.0552	2.205	0.3309	2.007	0.0595	0.802	0.0287	2.204	0.0705
		1.35	2.228	0.2169	0.937	0.0490	2.136	0.2883	1.964	0.0568	0.797	0.0271	2.208	0.0638
	0.80	1.00	2.161	0.2086	0.911	0.0443	2.178	0.3438	2.008	0.0567	0.806	0.0252	2.204	0.0705
		1.40	2.250	0.2350	0.926	0.0412	2.083	0.3050	1.974	0.0525	0.795	0.0264	2.208	0.0663

Table 9: ALs of Symmetric CRI of β for order restricted case.

			Non-informative Prior			Informative Prior		
n	τ_1	τ_2	90%	95%	99%	90%	95%	99%
40	0.60	0.80	1.573	1.763	2.037	0.842	0.989	1.234
		1.20	1.504	1.751	2.167	0.711	0.821	0.990
	0.65	0.85	1.631	1.846	2.146	0.827	0.971	1.211
		1.25	1.455	1.693	2.097	0.696	0.804	0.973
	0.70	0.90	1.656	1.905	2.257	0.814	0.956	1.193
		1.30	1.387	1.610	2.001	0.680	0.787	0.957
	0.75	0.95	1.611	1.893	2.328	0.793	0.930	1.159
		1.35	1.313	1.525	1.900	0.667	0.773	0.944
	0.80	1.00	1.533	1.817	2.323	0.772	0.906	1.127
		1.40	1.222	1.421	1.777	0.653	0.758	0.929
50	0.60	0.80	1.346	1.513	1.759	0.808	0.947	1.178
		1.20	1.359	1.581	1.950	0.662	0.762	0.917
	0.65	0.85	1.418	1.602	1.865	0.790	0.926	1.153
		1.25	1.310	1.521	1.880	0.642	0.741	0.895
	0.70	0.90	1.464	1.681	1.983	0.772	0.905	1.126
		1.30	1.237	1.433	1.775	0.631	0.729	0.884
	0.75	0.95	1.442	1.690	2.066	0.754	0.884	1.098
		1.35	1.153	1.336	1.661	0.611	0.707	0.862
	0.80	1.00	1.370	1.622	2.063	0.732	0.857	1.064
		1.40	1.059	1.229	1.538	0.596	0.691	0.847

Table 10: ALs of HPD CRI of β for order restricted case.

			Prior I			Prior II		
n	τ_1	τ_2	90%	95%	99%	90%	95%	99%
40	0.60	0.80	1.438	1.636	1.948	0.821	0.957	1.185
		1.20	1.427	1.659	2.062	0.690	0.794	0.955
	0.65	0.85	1.516	1.729	2.057	0.806	0.942	1.166
		1.25	1.371	1.595	1.991	0.673	0.775	0.936
	0.70	0.90	1.565	1.801	2.163	0.794	0.928	1.150
		1.30	1.291	1.507	1.894	0.657	0.758	0.920
	0.75	0.95	1.545	1.805	2.227	0.774	0.904	1.120
		1.35	1.208	1.415	1.791	0.643	0.743	0.906
	0.80	1.00	1.485	1.750	2.220	0.754	0.881	1.089
		1.40	1.110	1.306	1.670	0.628	0.727	0.891
50	0.60	0.80	1.219	1.395	1.677	0.783	0.911	1.128
		1.20	1.284	1.493	1.850	0.637	0.732	0.882
	0.65	0.85	1.310	1.493	1.783	0.765	0.892	1.106
		1.25	1.226	1.427	1.781	0.616	0.709	0.858
	0.70	0.90	1.379	1.584	1.899	0.749	0.874	1.082
		1.30	1.142	1.332	1.676	0.604	0.697	0.847
	0.75	0.95	1.382	1.610	1.973	0.732	0.855	1.057
		1.35	1.050	1.231	1.562	0.583	0.674	0.825
	0.80	1.00	1.326	1.561	1.970	0.712	0.830	1.025
		1.40	0.950	1.121	1.440	0.567	0.658	0.809

Table 11: ALs of Symmetric CRI of λ_1 for order restricted case.

			Prior I			Prior II		
n	τ_1	τ_2	90%	95%	99%	90%	95%	99%
40	0.60	0.80	1.107	1.291	1.604	0.672	0.800	1.052
		1.20	1.080	1.274	1.636	0.650	0.773	1.012
	0.65	0.85	1.036	1.215	1.525	0.642	0.764	1.005
		1.25	0.973	1.151	1.488	0.625	0.744	0.974
	0.70	0.90	0.945	1.117	1.426	0.609	0.726	0.954
		1.30	0.879	1.041	1.352	0.598	0.712	0.933
	0.75	0.95	0.850	1.010	1.313	0.585	0.697	0.915
		1.35	0.796	0.946	1.233	0.572	0.681	0.893
	0.80	1.00	0.767	0.914	1.199	0.558	0.665	0.874
		1.40	0.727	0.866	1.131	0.550	0.655	0.859
50	0.60	0.80	0.967	1.128	1.411	0.636	0.759	0.998
		1.20	0.993	1.171	1.506	0.606	0.721	0.945
	0.65	0.85	0.915	1.074	1.352	0.601	0.717	0.943
		1.25	0.893	1.057	1.365	0.579	0.689	0.904
	0.70	0.90	0.846	1.000	1.279	0.571	0.680	0.894
		1.30	0.799	0.948	1.233	0.550	0.655	0.859
	0.75	0.95	0.764	0.910	1.181	0.542	0.646	0.849
		1.35	0.721	0.857	1.119	0.528	0.629	0.825
	0.80	1.00	0.693	0.825	1.081	0.518	0.618	0.812
		1.40	0.653	0.777	1.017	0.507	0.604	0.792

Table 12: ALs of HPD CRI of λ_1 for order restricted case.

			Prior I			Prior II		
n	τ_1	τ_2	90%	95%	99%	90%	95%	99%
40	0.60	0.80	1.025	1.212	1.537	0.658	0.784	1.031
		1.20	1.035	1.227	1.584	0.636	0.757	0.992
	0.65	0.85	0.970	1.149	1.463	0.629	0.751	0.987
		1.25	0.935	1.111	1.444	0.612	0.729	0.955
	0.70	0.90	0.897	1.065	1.372	0.598	0.713	0.938
		1.30	0.845	1.006	1.314	0.586	0.698	0.916
	0.75	0.95	0.818	0.973	1.268	0.575	0.685	0.901
		1.35	0.765	0.913	1.199	0.561	0.668	0.877
	0.80	1.00	0.746	0.889	1.164	0.549	0.655	0.860
		1.40	0.696	0.834	1.098	0.539	0.642	0.843
50	0.60	0.80	0.899	1.063	1.353	0.623	0.744	0.979
		1.20	0.952	1.129	1.459	0.593	0.706	0.926
	0.65	0.85	0.861	1.017	1.298	0.590	0.704	0.927
		1.25	0.858	1.020	1.326	0.567	0.675	0.886
	0.70	0.90	0.805	0.955	1.231	0.560	0.668	0.879
		1.30	0.767	0.915	1.198	0.538	0.642	0.842
	0.75	0.95	0.737	0.877	1.142	0.533	0.635	0.835
		1.35	0.691	0.826	1.087	0.517	0.616	0.810
	0.80	1.00	0.674	0.804	1.052	0.510	0.608	0.799
		1.40	0.622	0.746	0.986	0.496	0.591	0.777

Table 13: ALs of Symmetric CRI of λ_2 for order restricted case.

			Prior I			Prior II		
n	τ_1	τ_2	90%	95%	99%	90%	95%	99%
40	0.60	0.80	2.393	2.919	4.097	0.887	1.057	1.392
		1.20	2.057	2.529	3.616	0.820	0.978	1.285
	0.65	0.85	2.402	2.927	4.085	0.886	1.057	1.391
		1.25	2.455	2.989	4.158	0.829	0.989	1.299
	0.70	0.90	2.264	2.769	3.890	0.885	1.056	1.388
		1.30	2.171	2.660	3.764	0.838	0.998	1.312
	0.75	0.95	2.555	3.111	4.320	0.887	1.059	1.391
		1.35	2.360	2.882	4.027	0.845	1.008	1.322
	0.80	1.00	2.628	3.198	4.428	0.890	1.062	1.396
		1.40	2.456	2.994	4.166	0.852	1.016	1.331
50	0.60	0.80	2.060	2.496	3.446	0.863	1.028	1.352
		1.20	1.798	2.193	3.068	0.790	0.942	1.237
	0.65	0.85	2.075	2.511	3.444	0.862	1.027	1.351
		1.25	1.880	2.288	3.180	0.800	0.954	1.251
	0.70	0.90	2.138	2.590	3.552	0.863	1.029	1.352
		1.30	1.977	2.403	3.324	0.810	0.966	1.266
	0.75	0.95	2.217	2.686	3.681	0.866	1.032	1.355
		1.35	2.050	2.488	3.429	0.818	0.976	1.277
	0.80	1.00	2.308	2.797	3.829	0.870	1.036	1.362
		1.40	2.122	2.575	3.538	0.827	0.986	1.289

Table 14: ALs of HPD CRI of λ_2 for order restricted case.

			Prior I			Prior II		
n	τ_1	τ_2	90%	95%	99%	90%	95%	99%
40	0.60	0.80	2.296	2.792	3.883	0.876	1.044	1.366
		1.20	1.957	2.399	3.399	0.808	0.962	1.257
	0.65	0.85	2.302	2.798	3.877	0.875	1.043	1.365
		1.25	2.348	2.854	3.948	0.815	0.971	1.270
	0.70	0.90	2.152	2.628	3.676	0.874	1.041	1.363
		1.30	2.065	2.526	3.551	0.823	0.980	1.281
	0.75	0.95	2.436	2.963	4.100	0.876	1.043	1.365
		1.35	2.243	2.736	3.810	0.829	0.988	1.290
	0.80	1.00	2.500	3.040	4.201	0.878	1.046	1.369
		1.40	2.333	2.842	3.945	0.836	0.995	1.298
50	0.60	0.80	1.996	2.413	3.303	0.852	1.014	1.327
		1.20	1.728	2.105	2.924	0.776	0.924	1.208
	0.65	0.85	2.008	2.428	3.311	0.850	1.013	1.325
		1.25	1.807	2.197	3.036	0.785	0.935	1.221
	0.70	0.90	2.063	2.496	3.409	0.851	1.013	1.326
		1.30	1.898	2.306	3.177	0.794	0.945	1.234
	0.75	0.95	2.134	2.584	3.529	0.853	1.016	1.328
		1.35	1.967	2.387	3.279	0.801	0.954	1.245
	0.80	1.00	2.217	2.685	3.668	0.856	1.020	1.334
		1.40	2.036	2.469	3.385	0.809	0.963	1.256

Once again in order restricted case, priors on β and λ_2 are 'almost' non-informative, whereas the prior on α is a non-informative in Prior I. In Prior II, the prior means of β, λ_1 and λ_2 are approximately 2.0, 0.8, and 2.2 respectively, whereas the prior variances are approximately 0.01 for all the parameters in both unrestricted and order restricted cases. Hence Prior II is an informative prior. For Prior I, we generate the failure times form the KHM with $\beta = 2$, $\lambda_1 = 1/1.2 \backsimeq 0.83$, and $\lambda_2 = 1/0.45 \backsimeq 2.22$. However, we generate β, λ_1, and λ_2 from their respective prior distribution and these generated β, λ_1, and λ_2 are used in KHM to generate failure times in case of Prior II. All the results are based on 5000 simulations and $M = 8000$. For different values of τ_1, τ_2, and n, AEs and MSEs of BEs for β, λ_1, and λ_2 are presented in Table 1 for unrestricted case and in Table 8 for order restricted case. ALs of symmetric and HPD CRIs of β, λ_1, and λ_2 are reported in Tables 2, 3, 4, 5, 6, and 7 for unrestricted case and in Tables 9, 10, 11,12, 13, and 14 for order restricted case. In all the calculations we discard those samples for which BE of any of the parameters is greater than ten times of its original value. We have noticed that for both values of n, there is only one sample for which BE of λ_1 is greater than 8.33 in case of unrestricted inference and for Prior I, when $\tau_1 = 0.6$ and $\tau_2 = 0.8$. We have also noticed that sometimes some of the points (less than 10 out of 8000) cover more than 99% of the weights in case of order restricted inference for the Prior II. All these points correspond to the outliers with respect to (14). Hence, we only accept a generated point x form (14), if $0.0003 < P(X \leq x) < 0.997$, where X has PDF (14).

As expected the MSEs of the estimates and ALs of the CRIs of the parameters for Prior II are less than that for Prior I. MSEs of all the unknown parameters decrease as n increases for both the priors and for both unrestricted and restricted cases. As τ_1 increases, MSEs of β and λ_1 decrease in unrestricted case. MSE of λ_2 decreases as τ_2 increases keeping τ_1 fixed under unrestricted framework. In the same case MSE of λ_1 also decreases with increase in τ_2. It is further noticed that ALs of symmetric and HPD CRI for all unknown parameters decrease as n increases keeping other parameters fixed. The MSEs of estimators of all unknown parameters decrease as τ_2 increases keeping other parameters fixed in the case of order restricted inference also. They also decrease as n increases. It is also observed

that MSEs of estimators of all unknown parameters are smaller in case of order restricted inference than those in the unrestricted case for both the priors.

5.2 Data Analysis

Table 15: Data for illustrative example.

Stress Level	Data							
1	0.1526	0.3381	0.3891	0.3936	0.4684	0.4716	0.4783	0.5575
	0.5685							
2	0.6009	0.6144	0.6276	0.6563	0.6566	0.6591	0.6629	0.6693
	0.6776	0.6948	0.6958	0.7089	0.7097	0.7113	0.7385	0.7679

Table 16: CRIs for unknown parameters for data in Table 15 under unrestricted priors.

		Prior I						Prior II					
		β		λ_1		λ_2		β		λ_1		λ_2	
Level	Type of CRI	LL	UL	LL	UL	LL	UL	LL	UL	LL	UL	LL	UL
90%	Symm. CRI	1.270	3.717	0.344	1.997	1.580	3.926	1.510	2.342	0.640	0.948	1.654	2.532
	HPD CRI	1.095	3.474	0.228	1.643	1.448	3.725	1.485	2.313	0.635	0.941	1.632	2.509
95%	Symm. CRI	1.120	4.038	0.290	2.382	1.437	4.249	1.441	2.444	0.613	0.980	1.579	2.627
	HPD CRI	1.053	3.891	0.195	2.023	1.328	4.053	1.413	2.400	0.609	0.974	1.562	2.605
99%	Symm. CRI	0.842	4.665	0.208	3.519	1.156	5.034	1.322	2.634	0.562	1.045	1.439	2.852
	HPD CRI	0.816	4.568	0.173	3.007	1.048	4.739	1.304	2.599	0.558	1.038	1.400	2.794

Table 17: CRIs for unknown parameters for data in Table 15 under order restricted priors.

		Prior I						Prior II					
		β		λ_1		λ_2		β		λ_1		λ_2	
Level	Type of CRI	LL	UL	LL	UL	LL	UL	LL	UL	LL	UL	LL	UL
90%	Symm. CRI	1.497	3.271	0.451	1.569	1.564	3.456	1.464	2.325	0.430	1.068	1.655	2.511
	HPD CRI	1.698	3.271	0.404	1.493	1.564	3.456	1.429	2.278	0.413	1.030	1.641	2.490
95%	Symm. CRI	1.329	3.271	0.384	1.863	1.433	3.845	1.419	2.422	0.385	1.141	1.597	2.620
	HPD CRI	1.497	3.271	0.345	1.631	1.352	3.610	1.381	2.340	0.378	1.120	1.575	2.574
99%	Symm. CRI	1.057	3.271	0.289	1.888	1.190	4.349	1.381	2.592	0.320	1.306	1.472	2.812
	HPD CRI	1.149	3.271	0.321	1.908	1.011	4.086	1.354	2.540	0.312	1.287	1.447	2.749

In this section we present a data analysis to illustrate the procedures described in Section 4. The data given in Table 15 is considered for this purpose. This data is artificially generated from KHM with $\beta = 2$, $\lambda_1 = 0.833$, $\lambda_2 = 2.222$, $\tau_1 = 0.6$, $\tau_2 = 0.8$, and $n = 40$. The priors assumptions are same as in Section 5.1. For Prior I, the estimates of β, λ_1, and λ_2 are 2.35, 0.93, and 2.61, respectively, in case of unrestricted inference, whereas in case of order restricted inference they are 2.49, 1.01, and 2.50, respectively. For Prior II, the estimates of β, λ_1, and λ_2 are 1.91, 0.78, and 2.08, respectively, in case of unrestricted inference,

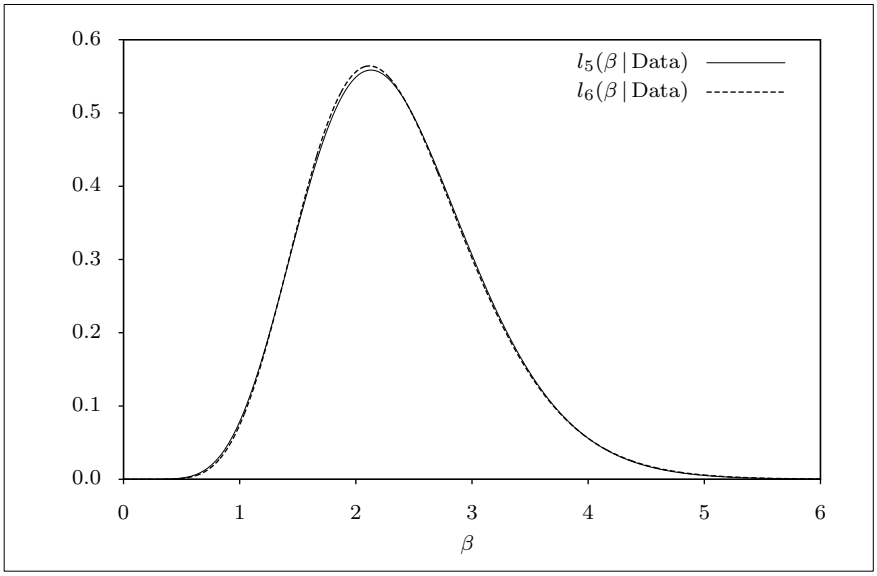


Figure 2: Gamma Approximation to $l_5(\beta \mid \text{Data})$ for Prior I.

whereas in case of order restricted inference they are 1.89, 0.72, and 2.06, respectively. Symmetric and HPD CRI of unknown parameters for unrestricted and order restricted priors are reported in Tables 16 and 17, respectively. Plot of marginal posterior density function of β and its gamma approximation is provided in Figure 2 which depicts that the approximation is quite nice at least for this data set.

6 Posterior Analysis under Other Censoring Schemes

Type-II Censoring Scheme

Based on the observed sample, the likelihood function is given in (5), where $\tau^* = t_{r:n}$, in Case (a), $n_1^* = 0$, $n_2^* = r$, in Case (b), $n_1^* = n_1$, $n_2^* = r - n_1$ and in Case (c), $n_1^* = r$, $n_2^* = 0$. $D_1(\beta)$ and $D_2(\beta)$ have the same expression as given in case of Type-I censoring.

Type-I Hybrid Censoring Scheme

Based on the data from Type-I HCS, the likelihood function is same as (5), where in Case (a), $n_1^* = 0$, $n_2^* = r$, in Case (b), $n_1^* = n_1$, $n_2^* = r - n_1$, in Case (c), $n_1^* = r$, $n_2^* = 0$, in Case (d), $n_1^* = 0$, $n_2^* = n_2$, in Case (e), $n_1^* = n_1$, $n_2^* = n_2$, and in Case (f), $n_1^* = n_1$, $n_2^* = 0$. Also in

the Cases (a)-(b), $\tau^* = t_{r:n}$, where for the rest of the cases $\tau^* = \tau_2$. $D_1(\beta)$ and $D_2(\beta)$ have the same expression as given in case of Type-I censoring.

Type-II Hybrid Censoring Scheme

Based on the observed sample from Type-II HCS, the likelihood function is given in (5), where in Case (a), $n_1^* = 0$, $n_2^* = r$, for Case (b), $n_1^* = n_1$, $n_2^* = r - n_1$, in Case (c), $n_1^* = 0$, $n_2^* = n_2$, for Case (d), $n_1^* = n_1$, $n_2^* = n_2$ and for Case (e), $n_1^* = n_1$, $n_2^* = 0$. $\tau^* = t_{r:n}$ for Cases (a) and (b), whereas for the rest of the cases $\tau^* = \tau_2$. $D_1(\beta)$ and $D_2(\beta)$ have the same expression as given in case of Type-I censoring.

Progressive Type-II Censoring Scheme

With the observed Progressive Type-II censoring data, the likelihood function is given by (5), where for Case (a), $n_1^* = 0$, $n_2^* = m$, for Case (b), $n_1^* = n_1$, $n_2^* = m - n_1$ and for Case (c) $n_1^* = m$, $n_2^* = 0$. For all the cases $\tau^* = t_{m:n}$, $D_1(\beta) = \sum_{k=1}^{n_1^*}(R_k+1)t_{k:n}^{\beta}+(n-n_1^*-\sum_{k=1}^{n^*} R_k)\tau_1^{\beta}$ and $D_2(\beta) = \sum_{k=n_1^*+1}^{m}(R_k+1)(t_{k:n}^{\beta} - \tau_1^{\beta})$.

In all the above cases, likelihood function are in the same form as Type-I censoring scheme and hence, the posterior density will also be in the same form as given in (6). In all these cases we will be able to compute the BE and construct the associated CRI for some function of unknown parameters exactly along the same line.

7 Conclusion

A simple SSLT has been considered under the Bayesian framework. It has been assumed that the lifetimes at each stress level have a Weibull distribution with common shape parameter and different scale parameters. Analysis has been performed under KHM assumption. We have discussed both unrestricted and order restricted inference of the unknown parameters. It is noticed that in most of the cases BE of parametric function cannot be obtained in closed form, when it exists. We have proposed algorithms based on the importance sampling to

compute BE and to construct associate CRI of parametric function. An extensive simulation has also been performed to judge the performance of the algorithms proposed. It is noticed that the proposed methods are working quite well for large values of n. For small values of n, MSEs of unknown parameters are quite large. It is also noticed that MSEs of BEs of unknown parameters are less in case of order restricted inference than those of unrestricted case. The proposed order restricted prior is a fairly general prior. It can be used for other lifetime distributions as well as for other censoring schemes also. It should be mentioned that the choice of proper priors is an important issue, which has not been pursued here and more work is needed in that direction.

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A Appendix

A.1 Integrability Conditions for Unrestricted Case

Note that $A_1(\beta) > 0$ and $A_2(\beta) > 0$ for all $\beta > 0$. Also $n_1^* + a_1 > 0$ and $n_2^* + a_2 > 0$. Now

$$\int_0^\infty \int_0^\infty \int_0^\infty l_2(\beta,\, \lambda_1,\, \lambda_2 \,|\, \text{Data}) d\lambda_1 d\lambda_2 d\beta \propto \int_0^\infty l_5(\beta \,|\, \text{Data}) d\beta,$$

where $l_5(\beta \,|\, \text{Data})$ is given in (10). Let us define

$$\tau_1^* = \begin{cases} \tau_1 & \text{if } n - n_1^* > 0 \\ t_{n:n} & \text{if } n - n_1^* = 0, \end{cases} \qquad \tau_2^* = \begin{cases} \tau_2 & \text{if } n - n^* > 0 \\ t_{n:n} & \text{if } n - n^* = 0. \end{cases}$$

Case I : $0 < \tau_1^* < \tau_2^* < 1$

In this case, $0 < t_{i:n} < 1$ for all $i = 1, 2, \ldots, n_1^* + n_2^*$ and hence, $A_1(\beta) \to b_1$, $A_2(\beta) \to b_2$ as $\beta \to \infty$. For some positive constants c_2 and c_3,

$$c_2 \int_0^\infty \beta^{n^* + a_3 - 1} e^{-(b_3 - c_1)\beta} d\beta \le \int_0^\infty l_5(\beta \,|\, \text{Data}) d\beta \le c_3 \int_0^\infty \beta^{n^* + a_3 - 1} e^{-(b_3 - c_1)\beta} d\beta.$$

Clearly $l_5(\beta \,|\, \text{Data})$ is integrable if $n^* + a_3 > 0$ and $b_3 - c_1 > 0$. As $c_1 = \sum_{i=1}^{n_1^*+n_2^*} \ln t_{i:n} < 0$, $l_2(\beta,\, \lambda_1,\, \lambda_2 \,|\, \text{Data})$ is integrable if $n_1^* + a_1 > 0$, $n_2^* + a_2 > 0$, $n^* + a_3 > 0$, and $b_3 > 0$.

Case II : $0 < \tau_1^* < 1 < \tau_2^*$

In this case, $0 < t_{i:n} < 1$ for $i = 1,\, 2,\, \ldots,\, n_1^*$ and hence, $A_1(\beta) \to b_1$ as $\beta \to \infty$. As $\beta \to \infty$, $\dfrac{A_2(\beta)}{\tau_2^{*\beta}} \to (n - n^*)$ if $n - n^* > 0$. If $n - n^* = 0$, $\dfrac{A_2(\beta)}{\tau_2^{*\beta}} \to 1$ as $\beta \to \infty$. Hence, for some positive constants c_2 and c_3,

$$c_2 \int_0^\infty \beta^{n^*+a_3-1} e^{-(b_3-c_1+(n_2^*+a_2)\ln\tau_2^*)\beta} d\beta \le \int_0^\infty l_5(\beta \,|\, \text{Data}) d\beta \le c_3 \int_0^\infty \beta^{n^*+a_3-1} e^{-(b_3-c_1+(n_2^*+a_2)\ln\tau_2^*)\beta} d\beta.$$

Clearly $l_5(\beta \,|\, \text{Data})$ is integrable if $n^* + a_3 > 0$ and $b_3 - c_1 + (n_2^* + a_2)\ln\tau_2^* > 0$. Now

$$b_3 - c_1 + (n_2^* + a_2)\ln\tau_2^* = b_3 - \sum_{i=1}^{n_1^*} \ln t_{i:n} + \sum_{i=n_1^*+1}^{n^*} (\ln\tau_2^* - \ln t_{i:n}) + a_2 \ln\tau_2^*.$$

As $0 < t_{i:n} < 1$ for $i = 1,\, 2,\, \ldots,\, n_1^*$ and $t_{i:n} \le \tau_2^*$ for $i = n_1^*+1,\, n_1^*+2,\, \ldots,\, n^*$, $\sum_{i=1}^{n_1^*} \ln t_{i:n} < 0$ and $\sum_{i=n_1^*+1}^{n^*} (\ln\tau^* - \ln t_{i:n}) > 0$. Therefore $l_2(\beta,\, \lambda_1,\, \lambda_2 \,|\, \text{Data})$ is integrable if $n_1^* + a_1 > 0$, $n_2^* + a_2 > 0$, $n^* + a_3 > 0$, and $b_3 > 0$.

Case III : $1 < \tau_1^* < \tau_2^*$

In this case,

$$\frac{A_1(\beta)}{\tau_1^*} \to \begin{cases} n - n_1^* & \text{if } n - n_1^* > 0 \\ 1 & \text{if } n - n_1^* = 0, \end{cases} \qquad \frac{A_2(\beta)}{\tau_2^*} \to \begin{cases} n - n^* & \text{if } n - n^* > 0 \\ 1 & \text{if } n - n^* = 0, \end{cases}$$

as $\beta \to \infty$. Hence, for some positive constants c_2 and c_3,

$$c_2 \int_0^\infty \beta^{n^*+a_3-1} e^{-(b_3-c_1+(n_1^*+a_1)\ln\tau_1^*+(n_2^*+a_2)\ln\tau_2^*)\beta} d\beta \le \int_0^\infty l_5(\beta \,|\, \text{Data}) d\beta \le c_3 \int_0^\infty \beta^{n^*+a_3-1} e^{-(b_3-c_1+(n_1^*+a_1)\ln\tau_1^*+(n_2^*+a_2)\ln\tau_2^*)\beta} d\beta.$$

Clearly $l_5(\beta \,|\, \text{Data})$ is integrable if $n^* + a_3 > 0$ and $b_3 - c_1 + (n_1^* + a_1) \ln \tau_1^* + (n_2^* + a_2) \ln \tau_2^* > 0$. Now

$$\begin{aligned} & b_3 - c_1 + (n_1^* + a_1) \ln \tau_1^* + (n_2^* + a_2) \ln \tau_2^* \\ & = b_3 + \sum_{i=1}^{n_1^*} (\ln \tau_1^* - \ln t_{i:n}) + \sum_{i=n_1^*+1}^{n^*} (\ln \tau_2^* - \ln t_{i:n}) + a_1 \ln \tau_1^* + a_2 \ln \tau_2^*. \end{aligned}$$

As $t_{i:n} \leq \tau_1^*$ for $i = 1, 2, \ldots, n_1^*$ and $t_{i:n} \leq \tau_2^*$ for $i = n_1^*+1, n_1^*+2, \ldots, n^*$, $\sum_{i=1}^{n_1^*} (\ln \tau_1^* - \ln t_{i:n}) > 0$ and $\sum_{i=n_1^*+1}^{n^*} (\ln \tau^* - \ln t_{i:n}) > 0$. Therefore $l_2(\beta, \lambda_1, \lambda_2 \,|\, \text{Data})$ is integrable if $n_1^* + a_1 > 0$, $n_2^* + a_2 > 0$, $n^* + a_3 > 0$, and $b_3 > 0$. Thus $l_2(\beta, \lambda_1, \lambda_2, \,|\, \text{Data})$ is integrable if proper priors are assumed on the unknown parameters for unrestricted inference case.

A.2 Integrability Conditions for Restricted Case

Note that $n^* + a_2 > 0$ and $\alpha D_1(\beta) + D_2(\beta) + b_2 > 0$ for all $\beta > 0$ and $\alpha \in (0, 1)$. Now

$$\int_0^\infty l_7(\alpha, \beta, \lambda_2 \,|\, \text{Data}) d\lambda_2 \propto \frac{\alpha^{n_1^*+a_4-1}(1-\alpha)^{b_4-1}\beta^{n^*+a_3-1}e^{-(b_3-c_1)\beta}}{\{\alpha D_1(\beta) + D_2(\beta) + b_2\}^{n^*+a_2}}. \tag{16}$$

Case I : $0 < \tau_2^* < 1$

For fixed $\alpha \in (0, 1)$, $\alpha D_1(\beta) + D_2(\beta) + b_2 \to b_2$ as $\beta \to \infty$, and hence, right hand side of (16) is integrable with respect to $\alpha \in (0, 1)$ and $\beta > 0$ if $n^* + a_4 > 0$, $b_4 > 0$, $n^* + a_3 > 0$, and $b_3 > 0$. Therefore $l_7(\alpha, \beta, \lambda_2 \,|\, \text{Data})$ is integrable if proper priors are assumed on the unknown parameters.

Case II : $\tau_2^* \geq 1$

For fixed $\alpha \in (0, 1)$, as $\beta \to \infty$,

$$\frac{\alpha D_1(\beta) + D_2(\beta) + b_2}{\tau_2^{*\beta}} \to \begin{cases} n - n^* & \text{if } n - n^* \geq 1 \\ 1 & \text{if } n - n^* = 0, \end{cases}$$

which is independent of $\alpha \in (0, 1)$. Hence, in this case also, right hand side of (16) is integrable with respect to $\alpha \in (0, 1)$ and $\beta > 0$ if $n^* + a_4 > 0$, $b_4 > 0$, $n^* + a_3 > 0$, and

$b_3 > 0$. Therefore $l_7(\alpha, \beta, \lambda_2 \,|\, \text{Data})$ is integrable under the same condition as above. Thus $l_7(\alpha, \beta, \lambda_2 \,|\, \text{Data})$ is a proper PDF whenever proper priors are assumed on the unknown parameters in the case of order restricted inference.

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