On Simple Step-stress Model for Two-Parameter Exponential Distribution*

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Abstract

In this paper, we consider the simple step-stress model for a two-parameter exponential distribution, when both the parameters are unknown and the data are Type-II censored. It is assumed that under two different stress levels, the scale parameter only changes but the location parameter remains unchanged. It is observed that the maximum likelihood estimators do not always exist. We obtain the maximum likelihood estimates of the unknown parameters whenever they exist. We provide the exact conditional distributions of the maximum likelihood estimators of the scale parameters. Since the construction of the exact confidence intervals is very difficult from the conditional distributions, we propose to use the observed Fisher Information matrix for this purpose. We have suggested to use bootstrap method for constructing confidence intervals. Bayes estimates and associated credible intervals are obtained using importance sampling technique. Extensive simulations are performed to compare the performances of the different confidence and credible intervals in terms of their coverage percentages and average lengths. The performances of the bootstrap confidence intervals are quite satisfactory even for small sample sizes.

Keywords: Step-stress model; Type-II censoring; two-parameter exponential distribution; maximum likelihood estimates; conditional moment generating function; confidence interval; Fisher information matrix; bootstrap confidence interval; Bayes estimate.

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1 Introduction

In many life-testing experiments it is quite difficult to obtain enough failure time data under normal condition. This is mainly due to the fact that many products now a days have very high reliability to maintain competitiveness. To overcome this problem, accelerated life testing (ALT) has been introduced to ensure a faster rate of failure, see for example Nelson [22, 23]; Meeker and Escobar [20]; and Bagdonavičius and Nikulin [3].

Step-stress testing is a special class of ALT, where two or more levels of stress factors are applied on the products. In this type of experiment, the products are first exposed to an initial stress level, say s_1, and then the stress levels are increased to $s_2 < \cdots < s_m$ gradually, at pre-fixed times $\tau_1 < \cdots < \tau_{m-1}$. The data collected from such an step-stress testing experiment, may then be extrapolated to estimate the underlying distribution of failure times under normal conditions. This process requires a model relating the level of stress and the failure time distributions. Several models are available in the literature for analyzing step-stress data. In this paper we mainly consider simple step-stress set up, *i.e.* only two stress factors are applied on the experimental units.

Let s_1 and s_2 be two constants, and consider the simple step-stress

$$s(t) = \begin{cases} s_1 \text{ if } 0 \leq t < \tau \\ s_2 \text{ if } t \geq \tau. \end{cases} \tag{1}$$

Suppose that $F_1(\cdot)$ and $F_2(\cdot)$ are the cumulative distribution functions (CDFs) of lifetimes under the constant stress level s_1 and s_2, respectively. Now we briefly discuss some of the very popular simple step-stress models.

The most popular one among the different models is the cumulative exposure model (CEM), which was originally introduced by Sedyakin [25], and latter generalized by Bagdonavičius [1], see also Nelson [22] in this respect. Under the CEM, the CDF of the lifetime of the experimental unit is given by

$$F_{\text{CE}}(t) = \begin{cases} F_1(t) & \text{if } 0 \leq t < \tau \\ F_2(t - \tau + \tau^*_{\text{CE}}) & \text{if } t \geq \tau \end{cases}$$

for the step-stress $s(\cdot)$, where $F_2(\tau^*_{\text{CE}}) = F_1(\tau)$.

Analysis of one-parameter exponential model in case of simple step-stress set up, under the CEM formulation has been performed quite extensively in the literature. The readers may refer to the work of Balakrishnan *et al.* [6], Miller and Nelson [21], Xiong [27] and the references cited therein. Recently, Balakrishnan [5] provided a synthesis of exact inferential results and other related issues for one-parameter exponential model under the CEM formulation. DeGroot and Goel [11], Dorp *et al.* [12], Lee and Pan [16, 17], Leu and Shen [18] provided the Bayesian inference of the unknown parameters of a simple step-stress model.

Another accelerated failure time (AFT) model for the time varying stresses was proposed by Bagdonavičius [1]. This model was build on the assumption of generalized Sedyakin (GS) model with CDFs of lifetimes under different stress levels satisfying $F_2(t) = F_1(rt)$ for some $r > 0$. Under this model the CDF of the lifetimes coming from step-stress $s(\cdot)$ is given by

$$F_{\text{AFT}}(t) = \begin{cases} F_1(t) & \text{if } 0 \le t < \tau \\ F_2(t - \tau + \tau^*_{\text{AFT}}) & \text{if } t \ge \tau, \end{cases}$$

where $\tau^*_{\text{AFT}} = r\tau$. Note that it is assumed here that switching of the stress level changes the scale parameter only. A generalization of this model has been suggested by Bagdonavičius and Nikulin, see for example [2] and [3], which assumes that the switch in stress levels not only change the scale parameter but also the shape parameter. Unfortunately analysis becomes quite difficult for this general model.

One of the widely used model describing the influence of covariates on the life time distribution is proportional hazards model (PHM) or Cox model, first introduced by Cox [9]. Bhattacharyya and Soejoeti [7] used the concept of PHM and provided the tempered failure rate model (TFRM), which assumes that the effect of switching the stress level from s_1 to s_2 is to multiply the hazard rate of the stress level s_1 by an unknown constant $\alpha > 0$, *i.e.*,

$$\lambda_{\text{TFRM}}(t) = \begin{cases} \lambda_1(t) & \text{if } 0 \le t < \tau \\ \alpha\lambda_1(t) & \text{if } t \ge \tau, \end{cases}$$

where $\lambda_1(\cdot)$ and $\lambda_{\text{TFRM}}(\cdot)$ are the hazard rates at the stress level s_1 and that under the simple step-stress $s(\cdot)$. Under this model the CDF of the lifetimes coming from step-stress $s(\cdot)$ is given by

$$F_{\text{TFRM}}(t) = \begin{cases} F_1(t) & \text{if } 0 \le t < \tau \\ 1 - \{1 - F_1(\tau)\}^{1-\alpha} \{1 - F_1(t)\}^{\alpha} & \text{if } t \ge \tau. \end{cases}$$

However, TFRM cannot be used to model the lifetimes of the product which has aging property. Note that the ratio of hazard rates under different stress levels are assumed to be constant in time in this model. One can think of generalization of this model in this respect taking increasing, decreasing, or cross-effects ratio of hazard rates, see Bagdonavičius and Nikulin [3]. It is worth mentioning here that models for step-stress life test were also discussed by Bagdonavičius and Nikulin [2], Bagdonavičius and Nikulin [3], Bagdonavičius *et al.* [4], and Gerville-Réach and Nikulin [15] under the general framework of accelerated life tests.

The main purpose of this study is to consider two-parameter exponential distribution for the same simple step-stress model under the CEM formulation. It is assumed that as the stress level changes from s_1 to s_2, the scale parameter of the exponential distribution changes from θ_1 to θ_2, but the location parameter μ remains unchanged. One of the possible justifications of the assumption of common location parameter is the presence of unknown calibration in the equipment used for measuring lifetimes. The data are assumed to be Type-II censored. It is observed that the maximum likelihood estimators (MLEs) of the unknown parameters do not always exist, whenever they exist, they can be obtained in closed form. We obtain the exact conditional distributions of the MLEs of the scale parameters. Since the conditional distributions of the MLEs of the scale parameters depend on the unknown location parameter μ, it is not possible to obtain the exact confidence intervals (CIs) of the scale parameters based on the exact conditional distributions. We propose to use the Fisher information matrix to construct the asymptotic CIs of the unknown scale parameters, assuming the location parameter to be known. We also propose to use the parametric bootstrap method for constructing CIs of the scale parameters, and it is very easy to implement it in practice.

We further consider Bayesian inference of the unknown parameters θ_1, θ_2 and μ. It is assumed that θ_2 has an inverted gamma prior, and α has a beta prior, where $\theta_1\alpha = \theta_2$.

The location parameter μ is assumed to have a non-informative prior. Based on the above priors the Bayes estimates and the associate credible intervals are obtained using importance sampling technique. Extensive simulations are performed to compare the performances of the different methods and the performances are quite satisfactory. One data set has been analyzed for illustrative purposes.

Rest of the paper is organized as follows. In Section 2 first we discuss the model formulation and then provide the MLEs of the three unknown parameters. The conditional distribution of the MLEs of the scale parameters are presented in Section 3. In Section 4 we discuss the construction of different CIs for the scale parameters. Bayesian inference of the model parameters is indicated in Section 5. Simulation results and a data analysis are provided in Section 6, and finally conclusion appear in Section 7.

2 Model Description and MLEs

2.1 Model Description

We consider a simple step-stress model, where n identical units are placed on a life-testing experiment at the initial stress level s_1. The stress level is changed to a higher level s_2 at a prefixed time τ. Further the experiment is terminated as soon as the rth ($r \leq n$ is a prefixed integer) failure occurs. The failure times $t_{1:n} < \cdots < t_{r:n}$ denote the observed data. The following cases may be observed:

Case-I: $t_{1:n} < \cdots < t_{r:n} < \tau$,

Case-II: $t_{1:n} < \cdots < t_{N:n} < \tau < t_{N+1:n} < \cdots < t_{r:n}$,

Case-III: $\tau < t_{1:n} < \cdots < t_{r:n}$,

where N is the number of failures at the stress level s_1. Note that for Case-I and Case-III, $N = r$ and $N = 0$, respectively. Moreover, we assume that the lifetime distributions at two different stress levels satisfy CEM assumptions.

We also assume that the lifetime distributions at the stress levels s_1 and s_2 are exponential with scale parameters θ_1 and θ_2 respectively and a common location parameter μ. Presence of the common location parameter is justified in view of the possible unknown bias in the lifetime measurement system. Note that if $\mu \geq \tau$, then there is no observation at the first stress level, and hence θ_1 is not estimable. Therefore, it is assumed that $\mu < \tau$. Then under the assumption of the CEM, the cumulative distribution function (CDF), $F_T(\cdot)$, of a life time of an item is given by

$$F_T(t) = \begin{cases} 1 - e^{-\frac{t-\mu}{\theta_1}} & \text{if } \mu < t \leq \tau \\ 1 - e^{-\frac{t-\tau}{\theta_2} - \frac{\tau-\mu}{\theta_1}} & \text{if } \tau < t < \infty, \end{cases} \tag{2}$$

and the corresponding probability density function (PDF), $f(t)$, is given by

$$f_T(t) = \begin{cases} \frac{1}{\theta_1} e^{-\frac{t-\mu}{\theta_1}} & \text{if } \mu < t \leq \tau \\ \frac{1}{\theta_2} e^{-\frac{t-\tau}{\theta_2} - \frac{\tau-\mu}{\theta_1}} & \text{if } \tau < t < \infty. \end{cases} \tag{3}$$

2.2 Likelihood Function and MLEs

In this section we consider the likelihood function of the observed data and obtain the MLEs of the unknown parameters. Then using (3), the likelihood of the observed data is given by

$$L(\mu, \theta_1, \theta_2) = \begin{cases} \dfrac{1}{\theta_2^r} \, e^{-\frac{n}{\theta_1}\tau + \frac{n}{\theta_1}\mu - \frac{D_2}{\theta_2}} & \text{if } N = 0 \\ \dfrac{1}{\theta_1^N \theta_2^{r-N}} e^{-\frac{D_1}{\theta_1} - \frac{n}{\theta_1}(t_{1:n}-\mu) - \frac{D_2}{\theta_2}} & \text{if } 1 \leq N \leq r-1 \\ \dfrac{1}{\theta_1^r} \, e^{-\frac{D_1}{\theta_1} - \frac{n}{\theta_1}(t_{1:n}-\mu)} & \text{if } N = r\,, \end{cases} \tag{4}$$

where $D_1 = \sum_{j=1}^{N} t_{j:n} + (n-N)m - nt_{1:n}$, $D_2 = \sum_{j=N+1}^{r} t_{j:n} + (n-r)t_{r:n} - (n-N)m$, and $m = \min\{\tau,\, t_{r:n}\}$. From the likelihood function in (4), it is clear that MLE of θ_1 does not exist when $N = 0$, and θ_2 is not estimable for $N = r$. For $1 \leq N \leq r-1$, MLEs of μ ,θ_1, and θ_2 exist and are

$$\widehat{\mu} = t_{1:n}, \qquad \widehat{\theta}_1 = \frac{D_1}{N}, \qquad \text{and} \qquad \widehat{\theta}_2 = \frac{D_2}{r-N} \tag{5}$$

respectively. Clearly these MLEs are conditional MLEs of μ, θ_1, and θ_2 conditioning on the event $1 \leq N \leq r-1$.

3 Conditional Distribution of MLEs

In this section we provide the marginal distribution of the MLEs conditioning on $1 \leq N \leq r-1$. It can be obtained by inverting the conditional moment generating functions (CMGF) as it was first suggested by Bartholmew [10]. For simplicity, let us denote the CMGFs of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ given $A = \{1 \leq N \leq r-1\}$ by

$$M_1(\omega|A) = E[e^{\omega \widehat{\theta}_1}|1 \leq N \leq r-1] \tag{6}$$

and

$$M_2(\omega|A) = E[e^{\omega \widehat{\theta}_2}|1 \leq N \leq r-1]$$

respectively. Note that CMGF in (6) can be written as

$$M_1(\omega|A) = \sum_{i=1}^{r-1} E[e^{\omega \widehat{\theta}_1}|N = i] \times P[N = i|1 \leq N \leq r-1]. \tag{7}$$

Now the number of the failures before time τ, viz., N is a non-negative random variable with probability mass function (PMF)

$$P[N = i] = \binom{n}{i} (1 - e^{-\frac{\tau-\mu}{\theta_1}})^i \, e^{-(n-i)\frac{\tau-\mu}{\theta_1}} = p_i \, (\text{say}) \quad \text{for} \quad i = 0, 1, \cdots, n,$$

so that for $i = 1, \ldots, r-1$

$$P[N = i|1 \leq N \leq r-1] = \frac{p_i}{\sum_{j=1}^{r-1} p_j}.$$

The exact derivations of $E[e^{\omega \widehat{\theta}_1}|N = j]$ for $j = 1, \cdots, r-1$ are provided in Appendix A. Using the inversion formula, the exact conditional distribution of $\widehat{\theta}_1$ can be obtained from CMGF and the corresponding probability density function is given in Theorem 3.1.

Theorem 3.1. The PDF of $\widehat{\theta}_1$ conditioning on $\{1 \leq N \leq r-1\}$ is given by

$$\begin{aligned} f_{\widehat{\theta}_1}(t) = {} & c_{10} \, f_4\,(t - \tau_{10}; \theta_1\,(n-1)) - d_{10}\, f_4\,(t; \theta_1\,(n-1)) \\ & + \sum_{i=2}^{r-1} \sum_{j=0}^{i-1} c_{ij} \, f_3 \left(t - \tau_{ij}; i-1, \frac{\theta_1}{i}, \frac{(n-j-1)\theta_1}{i(j+1)} \right) \\ & - \sum_{i=2}^{r-1} \sum_{j=0}^{i-1} d_{ij} \, f_3 \left(t; i-1, \frac{\theta_1}{i}, \frac{(n-j-1)\theta_1}{i(j+1)} \right), \end{aligned} \tag{8}$$

where

$$
\begin{aligned}
d_{ij} &= \tfrac{(-1)^{i-j-1}}{\sum_{k=1}^{r-1} p_k} \tbinom{n}{i} \tbinom{i}{j+1} e^{-\frac{n}{\theta_1}(\tau-\mu)}, \\
c_{ij} &= \tfrac{(-1)^{i-j-1}}{\sum_{k=1}^{r-1} p_k} \tbinom{n}{i} \tbinom{i}{j+1} e^{-\frac{n-j-1}{\theta_1}(\tau-\mu)}, \\
\tau_{ij} &= \tfrac{1}{i}(n-j-1)(\tau-\mu),
\end{aligned}
\tag{9}
$$

$$
f_3(t;\eta,\xi_1,\xi_2) = \frac{1}{\xi_2\ (1+\xi_1/\xi_2)^{\eta}}\, e^{t/\xi_2} \int_{\max\{0,(1/\xi_1+1/\xi_2)t\}}^{\infty} \frac{1}{\Gamma(\eta)} z^{\eta-1} e^{-z} \mathrm{d}z \quad \text{for } t \in \mathbb{R},
$$

and

$$
f_4(t;\xi) = \begin{cases} \frac{1}{\xi} e^{t/\xi} & \text{if t} \in (-\infty, 0) \\ 0 & \text{otherwise.} \end{cases}
$$

■

Proof: See Appendix A.

Similarly, inverting the CMGF $M_2(\omega|A)$ of $\widehat{\theta}_2$, conditional PDF of $\widehat{\theta}_2$ can be obtained as;

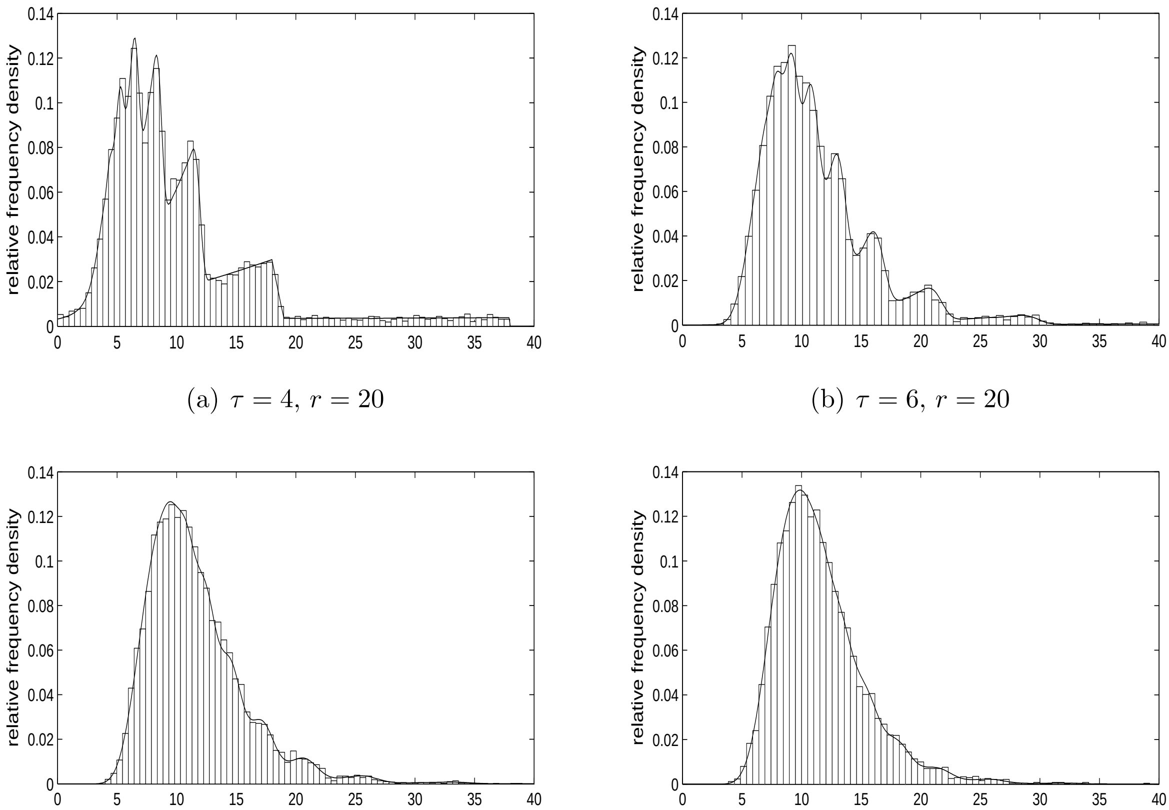
Theorem 3.2. The PDF of $\widehat{\theta}_2$ conditioning on $\{1 \le N \le r-1\}$ is given by

$$
f_{\widehat{\theta}_2}(t) = \sum_{i=1}^{r-1} c_i\, f_1\Big(t, r-i, \frac{\theta_2}{r-i}\Big),
$$

where $c_i = \dfrac{p_i}{\sum_{k=1}^{r-1} p_k}$ and $f_1(t,\eta,\xi) = \dfrac{1}{\xi^{\eta}\,\Gamma(\xi)}\, t^{\eta-1}\, e^{-t/\xi}$ if $t > 0$. ■

Proof: See Appendix A.

Since the shape of the conditional PDF of $\widehat{\theta}_1$ as given in Theorem 3.1 is difficult to analyze analytically, we provide the plots in Figure 1 of the PDFs of $\widehat{\theta}_1$ for $n = 20$, $\mu = 0$, $\theta_1 = 12$, $\theta_2 = 4.5$, and $r = 20$ (complete sample). We consider four different values of τ, *viz.*, 4, 6, 8, and 10. For comparison purposes, we have also generated samples from the same CEM model, and compute the MLEs of θ_1, θ_2 and μ, whenever they exist. We provide the histograms of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ based on 10000 replications along with the true conditional PDFs of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ respectively. It is clear that the true PDFs match very well with the corresponding histograms. The PDF plot of $\widehat{\theta}_2$ which is a mixture of gamma distributions is provided in Figure 2.



(a) $\tau = 4$, $r = 20$

(b) $\tau = 6$, $r = 20$

(c) $\tau = 8$, $r = 20$

(d) $\tau = 10$, $r = 20$

Figure 1: PDF-plot of $\widehat{\theta_1}$ for different values of τ and for $n = r = 20$, $\mu = 0$, $\theta_1 = 12$, and $\theta_2 = 4.5$.

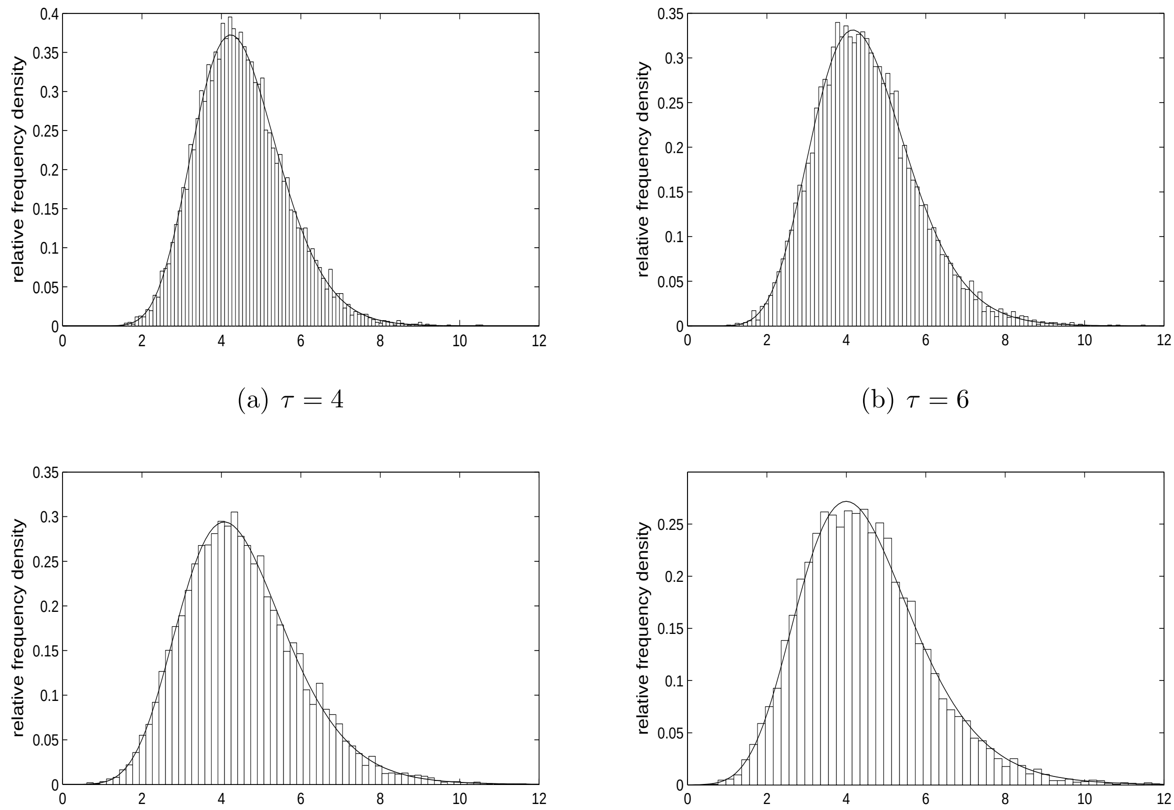
Remark 3.1. Note that the distribution of $\widehat{\mu}$ is same as conditional distribution of the first order statistic with a sample of size n form the two-parameter exponential distribution, conditioning on the event that there is at least one failure between the time μ and τ. Hence

$$E(\widehat{\mu}) = \frac{\mu}{q} + \frac{\theta_1}{n} - \frac{\tau(1-q)}{q},$$

where $q = 1 - e^{-n(\tau-\mu)/\theta_1}$ is the probability of getting at least one failure before the time τ. As MLEs do not exist if number of failure before time τ is zero, one should choose τ so that q is close to one. Hence using the above relation one can have a bias-reduced estimator of μ as

$$\widetilde{\mu} = \widehat{\mu} - \frac{\widehat{\theta_1}}{n}.$$

Remark 3.2. MLE of some parametric function, $g(\mu,\, \theta_1,\, \theta_2)$, can be obtained by replacing the parameters by their respectively MLEs, *i.e.* MLE of $g(\mu,\, \theta_1,\, \theta_2)$ is $g(\widehat{\mu},\, \widehat{\theta}_1,\, \widehat{\theta}_2)$. Now



(a) $\tau = 4$ (b) $\tau = 6$

(c) $\tau = 8$ (d) $\tau = 10$

Figure 2: PDF-plot of $\widehat{\theta}_2$ for different values of τ and for $n = 20$, $r = 20$, $\mu = 0$, $\theta_1 = 12$, and $\theta_2 = 4.5$

using the above estimator of μ, one will have a bias-reduced estimator of $g(\mu,\, \theta_1,\, \theta_2)$ as $\widetilde{g} = g(\widetilde{\mu},\, \widehat{\theta}_1, \widehat{\theta}_2)$. For example if $\eta_p = \mu - \theta_1 \ln(1-p)$, *i.e.*, p-th quantile of the lifetimes at the first stress level, then MLE of η_p is $\widehat{\eta}_p = \widehat{\mu} - \widehat{\theta} \ln(1-p)$ and a bias-reduced estimator is $\widetilde{\eta}_p = \widetilde{\mu} - \widehat{\theta} \ln(1-p)$.

4 Different types of confidence interval of $\widehat{\theta}_1$ and $\widehat{\theta}_2$

4.1 Asymptotic Confidence Interval

In the absence of a closed form of the conditional cumulative density functions of the parameter estimates $\widehat{\theta}_1$ and $\widehat{\theta}_2$, we cannot obtain the exact CIs. Because of the complicated nature of these integrals, we cannot consider the tail probabilities of $\widehat{\theta_1}$ and $\widehat{\theta_2}$ for the construction of exact CIs as in Chen and Bhattacharyya [8]. Moreover, it is observed empirically that $P_{\theta_1}(\widehat{\theta_1} > b)$ is not a monotone function of θ_1. Hence the construction of the exact confidence intervals become very difficult. Due to this reason, we proceed to obtain the asymptotic CIs of θ_1 and θ_2. We provide the elements of the Fisher information matrix. Though we have three parameters μ, θ_1, θ_2, we obtain the Fisher information matrix for θ_1 and θ_2 only, assuming μ is known and use the estimate of μ in the final expressions. We then use the asymptotic normality of the MLEs to construct asymptotic CIs of θ_1 and θ_2. For the purpose of comparison we also use parametric bootstrap methods (see Efron and Tibshirani [13] for details) to construct CIs for the two scale parameters.

Let $I(\theta_1, \theta_2) = (I_{ij}(\theta_1, \theta_2))$; $i, j = 1, 2$ denote the Fisher information matrix of θ_1 and θ_2, where

$$I_{11}(\theta_1, \theta_2) = E\big[-\tfrac{N}{\theta_1^2} + \tfrac{2D_1}{\theta_1^3}\big], \quad I_{12}(\theta_1, \theta_2) = 0,$$

$$I_{21}(\theta_1, \theta_2) = 0, \qquad I_{22}(\theta_1, \theta_2) = E\big[-\tfrac{r-N}{\theta_2^2} + \tfrac{2D_2}{\theta_2^3}\big].$$

The observed information matrix is

$$\begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} = \begin{bmatrix} \frac{N}{\widehat{\theta}_1^2} & 0 \\ 0 & \frac{r-N}{\widehat{\theta}_2^2} \end{bmatrix}.$$

The variance of $\widehat{\theta_1}$ and $\widehat{\theta_2}$ can be obtained through the observed information matrix as

$$V_1 = \frac{\widehat{\theta}_1^2}{N} \quad \text{and} \quad V_2 = \frac{\widehat{\theta}_2^2}{r-N}.$$

The asymptotic distributions of the pivotal quantities $\frac{\widehat{\theta}_1 - E(\widehat{\theta}_1)}{\sqrt{V1}}$ and $\frac{\widehat{\theta}_2 - E(\widehat{\theta}_2)}{\sqrt{V2}}$ may then be used to construct $100(1-\alpha)\%$ CIs for θ_1 and θ_2 respectively. The $100(1-\alpha)\%$ confidence interval for θ_1 and θ_2 are given by

$$\big[\widehat{\theta}_1 \pm z_{1-\frac{\alpha}{2}}\sqrt{V_1}\big] \quad \text{and} \quad \big[\widehat{\theta}_2 \pm z_{1-\frac{\alpha}{2}}\sqrt{V_2}\big].$$

4.2 Bootstrap Confidence Interval

In this subsection, we construct bootstrap CIs based on parametric bootstrap. Later we show that bootstrap CIs has better coverage probabilities than asymptotic CIs unless the sample size is quite large. Now we describe the algorithm to obtain bootstrap CIs for θ_1 and θ_2.

Parametric bootstrap:

1. Given τ, n and the original sample $t = (t_{1:n}, t_{2:n}, \cdots, t_{r:n})$ obtain $\widehat{\mu}$, $\widehat{\theta}_1$ and $\widehat{\theta}_2$, the MLEs of μ, θ_1, and θ_2 respectively.

2. Simulate a sample of size n from uniform $(0,1)$ distribution, denote the ordered sample as $U_{1:n}, U_{2:n}, \cdots, U_{n:n}$.

3. Find N, such that $U_{N:n} \le 1 - e^{-\frac{\tau-\widehat{\mu}}{\widehat{\theta}_1}} \le U_{N+1:n}$.

4. If $1 \le N \le r-1$, proceed to the next step. Otherwise go back to Step 2.

5. For $j \le N$, $T_{j:n} = \widehat{\mu} - \widehat{\theta}_1 \ln(1 - U_{j:n})$. For $j = N+1, \cdots, n$, $T_{j:n} = \tau - \frac{\widehat{\theta}_2}{\widehat{\theta}_1}(\tau - \widehat{\mu}) - \widehat{\theta}_2 \ln(1 - U_{j:n})$.

6. Compute the MLEs of θ_1 and θ_2 based on $T_{1:n}, T_{2:n}, \cdots, T_{r:n}$, say $\widehat{\theta}_1^{(1)}$ and $\widehat{\theta}_2^{(1)}$.

7. Repeat Steps 2-5 M times and obtain $\widehat{\theta}_1^{(1)}, \widehat{\theta}_2^{(1)}, \widehat{\theta}_1^{(2)}, \widehat{\theta}_2^{(2)}, \cdots, \widehat{\theta}_1^{(M)}, \widehat{\theta}_2^{(M)}$.

8. Arrange $\widehat{\theta}_1^{(1)}, \widehat{\theta}_1^{(2)}, \cdots, \widehat{\theta}_1^{(M)}$ in ascending order and obtain $\widehat{\theta}_1^{[1]}, \widehat{\theta}_1^{[2]}, \cdots, \widehat{\theta}_1^{[M]}$. Similarly, arrange $\widehat{\theta}_2^{(1)}, \widehat{\theta}_2^{(2)}, \cdots, \widehat{\theta}_2^{(M)}$ in ascending order and obtain $\widehat{\theta}_2^{[1]}, \widehat{\theta}_2^{[2]}, \cdots, \widehat{\theta}_2^{[M]}$. A two-sided $100(1-\alpha)\%$ bootstrap confidence interval of θ_i, $(i = 1, 2)$ is then given by

$$\left(\widehat{\theta}_i^{[\frac{\alpha}{2}M]}, \widehat{\theta}_i^{[(1-\frac{\alpha}{2})M]}\right),$$

   where $[x]$ denotes the largest integer less than or equal to x.

5 Bayesian Inference

As the conditional distribution of the MLEs of the unknown parameters are quite complicated, Bayesian analysis seems to be a natural choice. Also it is well known that the bootstrap CI of the thresh hold parameter μ does not work very well, see for example Shao and Tu ([26], Chapter 3), but a proper Bayesian credible interval(CRI) for μ can be obtained in a standard manner. In this paper we mainly consider the square error loss function, although any other loss functions can be considered in a similar fashion. Here we assume that the data are coming form the distribution as mentioned in (2). To proceed further, we need to make some prior assumptions on the unknown parameters. Note that the basic aim of the step-stress life tests is to get more failures at the higher stress level, hence it is reasonable to assume that $\theta_1 > \theta_2$. One of the prior assumption that supports $\theta_1 > \theta_2$ is $\theta_1 = \frac{\theta_2}{\alpha}$, where $0 < \alpha < 1$. Here we assume that θ_2 has an inverted gamma (IG) distribution with parameters $a > 0$, $b > 0$, α has a beta distribution with parameters $c > 0$, $d > 0$, and the location parameter μ has a non-informative prior over $(-\infty, \tau)$. The prior density for θ_2, α and μ are given by

$$\begin{aligned}
\pi_1(\theta_2) &= \frac{b^a}{\Gamma(a)} \frac{e^{-b/\theta_2}}{\theta_2^{a+1}} && \text{if } \theta_2 > 0, \\
\pi_2(\alpha) &= \frac{1}{B(c,d)}\, \alpha^{c-1}(1-\alpha)^{d-1} && \text{if } 0 < \alpha < 1, \\
\pi_3(\mu) &= 1 && \text{if } -\infty < \mu < \tau
\end{aligned}$$

respectively. We also assume that μ, α, and θ_2 are independently distributed. Likelihood function of the data for given (μ, α, θ_2) can be expressed as

$$l(\text{Data}|\mu,\, \alpha,\, \theta_2) \propto \begin{cases} \dfrac{\alpha^N}{\theta_2^r}\, e^{-\frac{1}{\theta_2}\{\alpha\, D_3 + D_2 - n\,\alpha\,\mu\}} & \text{if } \mu < t_{1:n} < \cdots < t_{N:n} < \tau \\ & \quad < t_{N+1:n} < \cdots < t_{r:n}, 1 \le N \le r-1 \\ \dfrac{\alpha^r}{\theta_2^r}\, e^{-\frac{1}{\theta_2}\{\alpha\, D_3 - n\,\alpha\,\mu\}} & \text{if } \mu < t_{1:n} < \cdots < t_{r:n} < \tau \\ \dfrac{1}{\theta_2^r}\, e^{-\frac{1}{\theta_2}\{n\,\tau\,\alpha + D_2 - n\,\alpha\,\mu\}} & \text{if } \mu < \tau < t_{1:n} < \cdots < t_{r:n}, \end{cases}$$

where $D_3 = D_1 + nt_{1:n}$. Hence for $0 < \alpha < 1, \theta_2 > 0$, the posterior density of the parameters given data can be written as

Case-I : $N = 0$

$$l(\mu,\, \alpha,\, \theta_2|\text{Data}) \propto \frac{1}{\theta_2^{r+a+1}}\, \alpha^{c-1}\, (1-\alpha)^{d-1}\, e^{-\frac{1}{\theta_2}\{n\,\alpha\,\tau + D_2 + b - n\,\alpha\,\mu\}} \text{ if } \mu < \tau.$$

Case-II : $N = 1, 2, \cdots, r$

$$l(\mu,\, \alpha,\, \theta_2|\text{Data}) \propto \frac{1}{\theta_2^{r+a+1}}\, \alpha^{N+c-1}\, (1-\alpha)^{d-1}\, e^{-\frac{1}{\theta_2}\{\alpha\, D_3 + D_2 + b - n\,\alpha\,\mu\}} \text{ if } \mu < t_{1:n}.$$

The Bayes estimate $\widehat{g}(\mu,\,\alpha,\,\theta_2)$ of some function, say $g(\mu,\,\alpha,\,\theta_2)$, with respect to the square error loss function is the posterior expectation of $g(\mu,\,\alpha,\,\theta_2)$, *i.e.*, it can be expressed as

$$\widehat{g}(\mu,\,\alpha,\,\theta_2) = \int_0^1\int_0^\infty\int_{-\infty}^{t_{1:n}} g(\mu,\,\alpha,\,\theta_2)\, l(\mu,\,\alpha,\,\theta_2|\text{Data}) d\mu\, d\theta_2\, d\alpha. \tag{10}$$

Unfortunately, (10) cannot be found explicitly for general function $g(\mu,\,\alpha,\,\theta_2)$. One can use numerical methods to compute (10). Alternatively, Lindley's approximation (see [19]) can be used to approximate (10). But credible interval (CRI) cannot be found by any of the above methods. Hence we propose importance sampling to compute the Bayes estimate and as well as to construct CRI. In Case-II, (10) can be written as

$$\widehat{g}(\mu,\,\alpha,\,\theta_2) = \frac{\displaystyle\int_0^1\int_0^\infty\int_{-\infty}^{t_{1:n}} g_1(\mu,\,\alpha,\,\theta_2) l_1(\alpha) l_2(\theta_2|\alpha) l_3(\mu|\alpha,\,\theta_2) d\mu\, d\theta_2\, d\alpha}{\displaystyle\int_0^1\int_0^\infty\int_{-\infty}^{t_{1:n}} g_2(\mu,\,\alpha,\,\theta_2) l_1(\alpha) l_2(\theta_2|\alpha) l_3(\mu|\alpha,\,\theta_2) d\mu\, d\theta_2\, d\alpha}, \tag{11}$$

where

$$g_1(\mu,\,\alpha,\,\theta_2) = \frac{g(\mu,\,\alpha,\,\theta_2)\,\alpha^{N+c-2}(1-\alpha)^{d-1}}{(D_1\alpha + D_2 + b)^{r+a-1}},$$

$$g_2(\mu,\,\alpha,\,\theta_2) = \frac{\alpha^{N+c-2}(1-\alpha)^{d-1}}{(D_1\alpha + D_2 + b)^{r+a-1}},$$

$$l_1(\alpha) = 1, \qquad 0 < \alpha < 1,$$

$$l_2(\theta_2|\alpha) = \frac{(D_1\alpha + D_2 + b)^{r+a-1}}{\Gamma(r+a-1)} \times \frac{e^{-(D_1\alpha + D_2 + b)/\theta_2}}{\theta_2^{r+a}}, \qquad \theta_2 > 0,$$

$$l_3(\mu|\alpha,\,\theta_2) = \frac{n\,\alpha}{\theta_2}\, e^{n\,\alpha(\mu - t_{1:n})/\theta_2}, \qquad \mu < t_{1:n}.$$

Note that $l_3(\mu|\alpha,\,\theta_2)$ has a closed and invertible distribution function, and hence one can easily draw sample from this density function. It may be noted that the above choice of g_1, g_2, l_1, l_2, and l_3 functions are not unique, but the performance based on them are quite satisfactory.

Algorithm for Case-II:

1. Generate α from U(0, 1).

2. Generate θ_2 from $\mathrm{IG}(r + a - 1,\, D_1\alpha + D_2 + b)$.

3. Generate μ from $l_3(\mu|\alpha,\, \theta_2)$.

4. Repeat steps 1-3 M times to obtain $\{(\mu_1,\, \alpha_1,\, \theta_{21}),\, \cdots,\, (\mu_M,\, \alpha_M,\, \theta_{2M})\}$.

5. Approximate (10) by

$$\widehat{g}(\mu,\, \alpha,\, \theta_2) = \frac{\displaystyle\sum_{i=0}^{M} g_1(\mu_i,\, \alpha_i,\, \theta_{2i})}{\displaystyle\sum_{i=0}^{M} g_2(\mu_i,\, \alpha_i,\, \theta_{2i})}$$

6. To find a $100\,(1-\gamma)\%$ CRI for $g(\mu,\, \alpha,\, \theta_2)$, arrange the $\{g(\mu_1,\, \alpha_1,\, \theta_{21}),\, \cdots,\, g(\mu_M,\, \alpha_M,\, \theta_{2M})\}$ to get $\left\{g^{(1)} < g^{(2)} < \cdots < g^{(M)}\right\}$. Arrange $\{g_2(\mu_1,\, \alpha_1,\, \theta_{21}),\, \cdots,\, g_2(\mu_M,\, \alpha_M,\, \theta_{2M})\}$ accordingly to get $\left\{g_2^{(1)},\, g_2^{(2)},\, \cdots,\, g_2^{(M)}\right\}$. Note that $g_2^{(i)}$'s are not ordered. Let

$$\bar{g}_2^{(i)} = \frac{g_2^{(i)}}{\displaystyle\sum_{j=1}^{M} g_2^{(j)}}.$$

A $100\,(1-\gamma)\%$ CRI is then given by $\left(g^{(i)},\, g^{(n+i)}\right)$, where

$$i \in \left\{ i \in \mathbb{N} : \sum_{j=1}^{i} \bar{g}_2^{(j)} \le 1 - \gamma \right\} \text{ and } \sum_{j=1}^{n+i} \bar{g}_2^{(j)} - \sum_{j=1}^{i} \bar{g}_2^{(j)} < 1 - \gamma \le \sum_{j=1}^{n+i+1} \bar{g}_2^{(j)} - \sum_{j=1}^{i} \bar{g}_2^{(j)}.$$

Therefore, $100\,(1-\gamma)\%$ highest posterior density (HPD) CRI of $g(\mu,\, \alpha,\, \theta_2)$ becomes $\left(g^{(i^*)},\, g^{(n+i^*)}\right)$, where $i^* \in \left\{ i \in \mathbb{N} : \sum_{j=1}^{i} \bar{g}_2^{(j)} \le 1 - \gamma \right\}$ satisfies

$$g^{(n+i^*)} - g^{(i^*)} \le g^{(n+i)} - g^{(i)}, \qquad \forall i \in \left\{ i \in \mathbb{N} : \sum_{j=1}^{i} \bar{g}_2^{(j)} \le 1 - \gamma \right\}.$$

For Case-I, (10) can be expressed in the same fashion as given in (11) with

$$\begin{aligned} g_1(\mu,\,\alpha,\,\theta_2) &= \frac{1}{\alpha}\, g(\mu,\,\alpha,\,\theta_2), \\ g_2(\mu,\,\alpha,\,\theta_2) &= \frac{1}{\alpha}, \\ l_1(\alpha) &= \frac{1}{B(c,\,d)}\alpha^{c-1}(1-\alpha)^{d-1}, \qquad 0<\alpha<1, \\ l_2(\theta_2|\alpha) &= \frac{(D_2+b)^{r+a-1}}{\Gamma(r+a-1)}\,\frac{e^{-(D_2+b)/\theta_2}}{\theta_2^{r+a}}, \qquad \theta_2>0, \\ l_3(\mu|\alpha,\,\theta_2) &= \frac{n\,\alpha}{\theta_2}\,e^{-n\,\alpha(\tau-\mu)/\theta_2}, \qquad \mu<\tau. \end{aligned}$$

Hence Bayes estimate and credible interval for $g(\mu,\,\alpha,\,\theta_2)$ can be found using importance sampling in Case-I in the same manner as in Case-II.

Table 1: Average estimates and the associated MSEs for MLE and bias-reduced estimator of μ.

			$\widehat{\mu}$		$\widetilde{\mu}$	
n	r	τ	AE	MSE	AE	MSE
30	30	3.0	0.395	0.319	0.017	0.199
		3.5	0.397	0.323	0.015	0.192
		4.0	0.397	0.326	0.006	0.191
30	20	3.0	0.395	0.319	0.017	0.204
		3.5	0.397	0.323	0.001	0.183
		4.0	0.397	0.326	0.007	0.178
40	40	3.0	0.299	0.186	0.003	0.102
		3.5	0.299	0.186	0.004	0.104
		4.0	0.299	0.186	-0.002	0.098
40	26	3.0	0.299	0.186	0.003	0.102
		3.5	0.299	0.186	0.004	0.104
		4.0	0.299	0.186	0.007	0.104
50	50	3.0	0.236	0.112	0.010	0.068
		3.5	0.236	0.112	-0.001	0.062
		4.0	0.236	0.112	-0.000	0.064
50	33	3.0	0.236	0.112	-0.004	0.063
		3.5	0.236	0.112	-0.002	0.060
		4.0	0.236	0.112	0.005	0.062

Table 2: Coverage percentages and average lengths of bootstrap and asymptotic confidence intervals along with average estimates and the associated MSEs for MLE of θ_1.

					95%				99%			
					BCI		ACI		BCI		ACI	
n	r	τ	AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL
30	30	3.0	11.962	38.754	92.72	21.315	84.42	20.647	98.38	35.269	91.48	27.135
		3.5	11.984	33.486	92.36	20.527	86.22	19.134	98.38	34.286	91.66	25.147
		4.0	11.925	27.370	93.26	19.402	86.62	17.937	98.56	31.918	93.30	23.573
30	20	3.0	11.962	38.754	93.22	21.977	86.18	21.007	98.34	36.415	92.08	27.607
		3.5	11.984	33.486	93.56	20.231	86.26	18.652	98.66	33.849	93.06	24.513
		4.0	11.796	22.487	93.12	19.146	87.70	17.515	98.48	31.550	92.98	23.019
40	40	3.0	12.010	27.018	94.02	19.224	87.54	17.418	98.74	32.005	93.52	22.892
		3.5	11.982	22.815	93.96	17.491	88.72	15.899	98.70	28.649	94.10	20.894
		4.0	11.767	15.041	93.82	15.956	89.36	14.628	98.82	25.159	94.58	19.224
40	26	3.0	12.010	27.018	93.52	19.110	87.80	17.292	98.92	32.063	93.04	22.726
		3.5	11.982	22.815	93.54	17.439	88.28	15.782	98.72	28.653	93.66	20.741
		4.0	11.793	17.007	94.02	16.074	89.18	14.747	98.82	25.389	94.58	19.381
50	50	3.0	11.993	18.326	94.10	16.599	89.28	15.084	98.88	26.894	94.02	19.824
		3.5	11.968	14.545	94.08	15.250	90.18	13.964	98.72	23.995	94.78	18.351
		4.0	11.795	11.707	93.34	13.876	89.72	12.922	98.66	20.947	94.82	16.983
50	33	3.0	11.993	18.326	93.62	16.589	88.30	15.039	98.84	26.954	93.66	19.764
		3.5	11.968	14.545	94.22	15.293	90.10	13.994	98.90	24.077	94.96	18.391
		4.0	11.900	12.578	93.58	14.107	89.94	13.114	98.84	21.378	94.88	17.235

6 Simulation Results and Data Analysis

To evaluate the performance of the CIs and CRIs we conduct a few simulation studies to obtain the coverage percentage (CP) and the average length (AL) of the CIs and CRIs of the different parameters of interest and the median of the first stress level. All the results are based on 5000 simulations with $\mu = 0$, $\theta_1 = 12$, $\theta_2 = 4.5$, $M = 3000$. We consider different values of n, *viz.*, 30, 40, and 50 and different values for τ, *viz.*, 3.0, 3.5, and 4.0. For each value of n, we consider $r = n$ and $r = 0.65\,n$. Here we take non-informative priors on all the parameters, *i.e.*, $a = 0$, $b = 0$, $c = 1$, $d = 1$ for the Bayesian inference, so that this result can be compared with that of frequentist approach. Note that Bayes estimate of μ and θ_1 exist if $N - 1 > 0$ and $r - 2 > 0$, that of θ_2 exists if $N > 0$ and $r - 2 > 0$. Also note that we discard those samples for which the Bayes estimate of θ_1 and θ_2 exceed 100 and those samples for which Bayes estimate of μ and median of the first stress level either less than

Table 3: Coverage percentages and average lengths of bootstrap and asymptotic confidence intervals along with the average estimates and the associated MSEs for MLE of θ_2.

					95%				99%			
					BCI		ACI		BCI		ACI	
n	r	τ	AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL
30	30	3.0	4.484	0.841	94.54	3.687	93.24	3.664	98.24	4.885	96.92	4.815
		3.5	4.482	0.883	94.24	3.753	93.00	3.726	98.02	4.977	96.52	4.896
		4.0	4.493	0.939	94.16	3.839	92.66	3.812	97.92	5.072	96.46	5.010
30	20	3.0	4.499	1.458	92.78	5.006	90.88	4.893	97.50	6.761	94.98	6.431
		3.5	4.498	1.621	93.46	5.253	91.06	5.091	97.50	7.152	95.08	6.691
		4.0	4.533	1.907	93.80	5.547	91.44	5.340	97.94	7.588	95.10	7.018
40	40	3.0	4.483	0.636	93.88	3.162	92.64	3.151	98.24	4.184	96.94	4.141
		3.5	4.482	0.660	94.42	3.257	93.46	3.241	98.48	4.309	97.40	4.259
		4.0	4.495	0.683	94.64	3.319	93.82	3.306	98.48	4.379	97.54	4.345
40	26	3.0	4.492	1.144	93.60	4.362	91.94	4.289	97.90	5.843	96.10	5.636
		3.5	4.490	1.243	93.28	4.583	91.22	4.476	97.46	6.181	95.62	5.882
		4.0	4.507	1.478	93.16	4.832	91.38	4.692	97.68	6.535	95.30	6.167
50	50	3.0	4.481	0.517	93.94	2.843	93.38	2.833	98.34	3.756	97.56	3.723
		3.5	4.480	0.537	94.04	2.903	93.34	2.891	98.32	3.837	97.46	3.799
		4.0	4.487	0.561	94.24	2.961	93.26	2.951	98.14	3.898	97.40	3.878
50	33	3.0	4.485	0.906	94.08	3.838	92.70	3.791	98.10	5.111	96.70	4.983
		3.5	4.485	0.979	93.98	3.982	92.42	3.920	98.26	5.326	96.40	5.152
		4.0	4.485	1.097	93.46	4.184	92.08	4.101	97.98	5.596	96.08	5.389

-100 or greater than 100. The results are provided in Tables 2-8.

From Tables 2 and 3, we see that the bootstrap CIs perform better than asymptotic CIs in terms of coverage percentage. For fixed n and r as the value of τ increases, performance of CIs for θ_1 improves on the account of availability of more data points and as expected, that of θ_2 deteriorates, but very marginally. Performance of Bayesian CRIs are quite satisfactory (see Tables 5-8). It is noticed that for fixed n and r as τ increases the performance of Bayes estimator and CRI of μ and θ_1 get improved, whereas performance of that of θ_2 get deteriorated, in the sense that MSE of the estimator and AL of corresponding CRI increase. Again for fixed τ, as n increases performance of estimator of all the parameters and all CRIs improve. As r increases the performance of all the estimators increases for fixed n and τ. We have also noticed that though the CP of the Bayesian CRIs are better than that of classical CIs, but the AL of classical CIs are lesser than that of Bayesian CRIs.

Table 4: Coverage percentages and average lengths of bootstrap confidence intervals of $\eta_{0.5}$ along with the average estimates and the associated MSEs of $\widetilde{\eta}_{0.5}$ as an estimator of $\eta_{0.5}$.

					95%		99%	
n	r	τ	AE	MSE	CP	AL	CP	AL
30	30	3.0	8.116	14.353	92.52	14.349	98.84	24.171
		3.5	8.124	12.938	92.42	13.413	98.34	22.492
		4.0	8.174	10.834	93.20	12.683	98.58	21.044
30	20	3.0	8.094	14.071	92.36	14.332	98.44	24.109
		3.5	8.187	13.323	92.72	13.549	98.64	22.723
		4.0	8.186	9.070	93.30	12.655	98.70	21.049
40	40	3.0	8.233	9.976	93.36	12.622	98.52	20.960
		3.5	8.225	9.197	93.14	11.590	98.52	18.757
		4.0	8.219	8.136	93.32	10.779	98.14	16.956
40	26	3.0	8.220	11.055	93.64	12.609	98.54	20.946
		3.5	8.172	8.165	93.36	11.465	98.60	18.572
		4.0	8.226	7.254	93.70	10.756	98.76	16.950
50	50	3.0	8.155	8.208	93.70	10.890	98.62	17.315
		3.5	8.247	6.626	93.92	10.155	98.78	15.759
		4.0	8.263	5.418	94.26	9.461	98.96	14.314
50	33	3.0	8.272	7.680	93.66	11.103	99.00	17.734
		3.5	8.234	6.309	93.78	10.125	98.76	15.658
		4.0	8.177	5.705	93.92	9.334	99.04	14.126

Next we provide a data analysis to illustrate the procedures described in sections 2, 4, and 5. A artificial data is generated from the CEM given in (2) with $n = 30$, $\mu = 10.0$, $\theta_1 = e^{2.5}$, $\theta_2 = e^{1.5}$, and $\tau = 14.5$ and is given in Table 9. Based on the assumption that the data given in Table 9 is coming from exponential CEM, MLE of all the three parameters can be found using (5) and Bayes estimates can be found using the algorithm described in section 5. Here also we consider two values of r, *viz.*, 30 and 20. For $r = 30$, MLE of μ, θ_1, θ_2 and median of the first stress level are 10.05, 17.22, 3.50, and 21.99 and Bayes estimate of that are 9.93, 8.52, 5.58, and 23.08 respectively. For $r = 20$, MLEs are 10.05, 17.21, 3.69, 21.99 and Bayes estimates are 9.89, 11.25, 7.37, and 23.37 respectively. Bias-reduced estimates of μ and median of the first stress level are same for both values of r and they are 9.48 and 21.40 respectively. Asymptotic and bootstrap CI, percentile and HPD CRI are also computed and reported in Table 10.

Table 5: Coverage percentages and average lengths of different credible intervals along with the average estimates and the associated MSEs for Bayes estimates of μ.

					Per. CRI				HPD CRI				% of
					95%		99%		95%		99%		samples
n	r	τ	AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL	discarded
30	30	3.0	-0.163	1.047	95.58	2.456	99.10	5.190	95.44	1.773	99.18	3.827	0.000
		3.5	-0.114	0.575	95.46	2.163	98.98	4.168	95.50	1.621	99.00	3.124	0.000
		4.0	-0.090	0.385	95.46	1.999	98.94	3.619	95.44	1.528	98.98	2.780	0.000
30	20	3.0	-0.182	2.200	95.18	2.510	99.04	5.454	95.62	1.795	99.42	3.913	0.020
		3.5	-0.106	0.366	95.24	2.141	99.12	3.826	95.72	1.614	99.18	3.028	0.000
		4.0	-0.076	0.289	95.16	2.006	99.04	3.473	95.50	1.536	99.04	2.745	0.000
40	40	3.0	-0.051	0.130	95.38	1.471	98.94	2.499	94.86	1.132	99.04	2.015	0.000
		3.5	-0.041	0.130	95.36	1.398	98.98	2.328	95.12	1.087	98.98	1.884	0.000
		4.0	-0.031	0.110	95.26	1.335	98.90	2.115	94.82	1.048	99.14	1.766	0.000
40	26	3.0	-0.065	0.178	95.62	1.507	99.24	2.593	95.88	1.154	99.26	2.076	0.000
		3.5	-0.049	0.107	95.34	1.403	99.16	2.276	95.78	1.093	99.24	1.879	0.000
		4.0	-0.039	0.099	95.32	1.342	99.24	2.127	95.48	1.054	99.20	1.776	0.000
50	50	3.0	-0.036	0.076	94.86	1.087	99.00	1.750	94.92	0.851	98.98	1.455	0.000
		3.5	-0.028	0.065	94.94	1.042	98.98	1.632	94.80	0.823	98.98	1.370	0.000
		4.0	-0.024	0.063	94.90	1.017	98.94	1.574	94.80	0.808	99.04	1.328	0.000
50	33	3.0	-0.031	0.073	95.30	1.084	98.78	1.732	94.72	0.850	98.84	1.439	0.000
		3.5	-0.024	0.070	94.94	1.043	98.74	1.634	94.74	0.823	98.66	1.371	0.000
		4.0	-0.021	0.067	94.80	1.016	98.70	1.571	94.86	0.807	98.78	1.326	0.000

7 Conclusion

The two-parameter exponential distribution has been considered in a simple step-stress model. Presence of the common location parameter is justified in the view of possibility of an unknown bias in the life-time experiment data. We obtain the exact distributions of the MLEs of the scale parameters at the two stress levels. The exact confidence limits of the scale parameters are difficult to obtain, due to the complicated nature of the model. We have proposed to use asymptotic and parametric bootstrap CIs, and the performance of the later is better. We have further proposed Bayesian inference of the unknown parameters under fairly general prior assumptions, and we obtained the Bayes estimates and the associated credible intervals using importance sampling technique. The proposed Bayes estimates and the credible intervals perform quite well.

Table 6: Coverage percentages and average lengths of different credible intervals along with the average estimates and the associated MSEs for Bayes estimates of θ_1.

					Per. CRI				HPD CRI				% of
					95%		99%		95%		99%		samples
n	r	τ	AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL	discarded
30	30	3.0	15.331	108.646	96.50	32.775	99.66	58.700	94.88	26.657	99.00	47.595	0.339
		3.5	14.639	84.449	95.70	27.737	99.30	46.712	94.76	23.348	98.98	39.176	0.140
		4.0	14.092	48.497	95.26	23.792	99.24	37.668	94.44	20.657	98.68	32.754	0.120
30	20	3.0	15.300	95.726	97.04	32.054	99.58	57.465	95.76	26.337	99.24	46.534	0.379
		3.5	14.733	72.332	96.70	27.527	99.50	46.018	95.90	23.301	99.22	38.804	0.100
		4.0	14.203	49.417	96.28	23.967	99.54	38.256	95.50	20.796	99.24	33.188	0.080
40	40	3.0	14.016	52.416	95.22	23.452	99.36	37.310	94.08	20.356	99.00	32.333	0.080
		3.5	13.570	36.284	95.00	20.248	99.38	30.893	94.54	18.025	99.08	27.512	0.080
		4.0	13.294	28.380	95.28	18.177	99.36	26.995	94.84	16.438	98.90	24.450	0.020
40	26	3.0	14.038	49.181	95.98	23.197	99.46	36.772	95.02	20.198	99.30	31.965	0.180
		3.5	13.737	37.947	95.90	20.502	99.22	31.311	94.86	18.237	99.12	27.884	0.040
		4.0	13.410	27.432	95.84	18.223	99.38	27.046	95.02	16.510	99.18	24.518	0.000
50	50	3.0	13.441	31.436	94.92	18.946	99.16	28.454	93.76	17.033	99.00	25.582	0.020
		3.5	13.144	22.906	95.00	16.741	99.02	24.440	93.96	15.319	98.96	22.394	0.000
		4.0	12.960	18.053	95.12	15.259	99.04	21.903	94.32	14.126	98.86	20.308	0.000
50	33	3.0	13.435	32.587	95.66	18.839	99.10	28.330	95.04	16.945	99.06	25.469	0.020
		3.5	13.117	21.933	95.78	16.580	99.26	24.189	94.60	15.180	99.10	22.169	0.020
		4.0	12.946	17.007	95.76	15.131	99.18	21.737	94.92	14.011	99.12	20.127	0.000

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Appendix: Lemmas and Proofs of the Theorems

Lemma .1. Let $X_{1:n} < \cdots < X_{n:n}$ be the order statistics of a random sample of size n from a continuous distribution with PDF $f(x)$. Let D denote the number of order statistics less than or equal to some pre-fixed number τ, such that $F(\tau) > 0$, where $F(.)$ is the distribution function of $f(.)$. The conditional joint PDF of $X_{1:n}, \cdots, X_{D:n}$ given that $D = j$ is identical with the joint PDF of all order statistics of size j from the right truncated density function

$$f_*(t) = \begin{cases} \frac{f(t)}{F(\tau)} & \text{for } t < \tau \\ 0 & \text{otherwise} \end{cases}.$$

Table 7: Coverage percentages and average lengths of different credible intervals along with the average estimates and the associated MSEs for Bayes estimates of θ_2.

					Per. CRI				HPD CRI				% of
					95%		99%		95%		99%		samples
n	r	τ	AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL	discarded
30	30	3.0	4.830	1.000	95.12	3.975	98.90	5.416	95.44	3.827	99.12	5.201	0.000
		3.5	4.860	1.059	95.14	4.088	99.06	5.569	95.74	3.932	99.18	5.346	0.000
		4.0	4.837	1.076	95.54	4.159	99.16	5.667	95.60	3.995	99.26	5.436	0.000
30	20	3.0	5.040	1.849	95.38	5.462	99.24	7.579	96.16	5.166	99.54	7.165	0.000
		3.5	5.045	1.877	96.04	5.668	99.40	7.867	96.64	5.349	99.62	7.426	0.000
		4.0	5.107	2.085	95.76	5.929	99.24	8.226	96.58	5.589	99.54	7.758	0.000
40	40	3.0	4.757	0.754	94.84	3.399	99.02	4.590	95.18	3.296	99.00	4.439	0.000
		3.5	4.775	0.802	94.64	3.491	99.00	4.715	95.08	3.382	99.00	4.558	0.000
		4.0	4.792	0.846	94.66	3.583	98.96	4.843	95.04	3.468	99.12	4.679	0.000
40	26	3.0	4.940	1.405	95.52	4.750	99.20	6.509	95.98	4.535	99.28	6.210	0.000
		3.5	4.977	1.517	95.58	4.970	99.26	6.810	96.10	4.736	99.34	6.490	0.000
		4.0	5.024	1.666	95.48	5.211	99.16	7.142	96.02	4.959	99.44	6.797	0.000
50	50	3.0	4.703	0.586	95.06	3.012	99.08	4.046	95.56	2.932	99.16	3.929	0.000
		3.5	4.716	0.618	94.98	3.090	99.12	4.153	95.60	3.006	99.18	4.030	0.000
		4.0	4.728	0.651	95.00	3.169	99.12	4.263	95.64	3.080	99.12	4.134	0.000
50	33	3.0	4.870	1.104	95.04	4.162	99.00	5.654	95.76	4.003	99.10	5.433	0.000
		3.5	4.907	1.205	95.20	4.360	98.96	5.925	95.92	4.186	99.00	5.686	0.000
		4.0	4.941	1.313	95.20	4.566	99.02	6.209	96.00	4.377	99.00	5.953	0.000

PROOF: See [6]

Lemma .2. Let $X_{1:n} < \cdots < X_{n:n}$ be the order statistics of a random sample of size n from a continuous distribution with PDF $f(x)$. Let D denote the number of order statistics less than or equal to some pre-fixed number τ, such that $F(\tau) > 0$, where $F(.)$ is the distribution function of $f(.)$. The conditional joint PDF of $X_{D+1:n}, \cdots, X_{n:n}$ given that $D = j$ is identical with the joint PDF of all order statistics of size $n-j$ from the left truncated density function

$$f_{**}(t) = \begin{cases} \frac{f(t)}{1-F(\tau)} & \text{for } \tau < t \\ 0 & \text{otherwise} \end{cases}.$$

PROOF: This can be proved following the same way of the prove of Lemma .1

Lemma .3. Let X be a Gamma(α, λ) random variable having the PDF

$$f_1(x;\, \alpha,\, \lambda) = \begin{cases} \frac{1}{\lambda^\alpha\, \Gamma(\alpha)}\, x^{\alpha-1}\, e^{-x/\lambda} & \text{for } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Table 8: Coverage percentages and average lengths of different credible intervals along with the average estimates and the associated MSEs for Bayes estimates of the median of the first stress level.

					Per. CRI				HPD CRI				% of
					95%		99%		95%		99%		samples
n	r	τ	AE	MSE	CP	AL	CP	AL	CP	AL	CP	AL	discarded
30	30	3.0	10.522	48.677	96.44	21.803	99.60	39.388	95.10	17.746	99.18	31.805	0.319
		3.5	10.062	37.875	95.86	18.509	99.24	31.308	94.78	15.582	99.06	26.239	0.120
		4.0	9.741	24.559	95.82	16.037	99.22	25.578	94.54	13.924	98.72	22.173	0.040
30	20	3.0	10.490	42.086	96.76	21.276	99.56	38.104	95.64	17.499	99.14	30.899	0.379
		3.5	10.115	31.838	96.68	18.289	99.44	30.562	95.70	15.502	99.22	25.828	0.100
		4.0	9.810	24.366	96.74	16.108	99.44	25.829	95.60	13.974	99.24	22.394	0.020
40	40	3.0	9.708	26.357	95.34	15.918	99.42	25.432	94.54	13.792	98.90	22.015	0.020
		3.5	9.409	19.095	95.20	13.776	99.32	21.176	94.26	12.238	99.02	18.788	0.020
		4.0	9.200	13.898	95.46	12.290	99.22	18.308	94.74	11.105	98.98	16.571	0.000
40	26	3.0	9.741	26.678	96.00	15.849	99.48	25.395	95.22	13.765	99.36	21.993	0.080
		3.5	9.506	19.604	95.92	13.908	99.22	21.371	94.88	12.350	99.08	18.960	0.000
		4.0	9.256	12.392	95.86	12.256	99.36	18.195	95.18	11.115	99.16	16.502	0.000
50	50	3.0	9.282	14.385	95.06	12.804	99.16	19.223	93.82	11.518	99.02	17.296	0.020
		3.5	9.083	10.457	95.42	11.322	98.92	16.529	93.88	10.367	98.82	15.151	0.000
		4.0	8.959	8.250	95.04	10.328	98.98	14.827	94.30	9.567	98.86	13.751	0.000
50	33	3.0	9.298	16.137	95.62	12.794	99.02	19.308	95.06	11.502	99.04	17.331	0.000
		3.5	9.068	10.032	95.90	11.212	99.22	16.363	94.78	10.270	99.08	15.001	0.020
		4.0	8.953	7.806	95.84	10.240	99.18	14.715	95.14	9.487	99.10	13.631	0.000

Then for any arbitrary constant A, the MGF of $A + X$ is given by

$$M_{A+X} = e^{A\,\omega}\,(1 - \lambda\,\omega)^{-\alpha} \text{ for } \omega < \tfrac{1}{\lambda}$$

Proof: It can be proved by simple integration and hence the proof is omitted.

Lemma .4. Let X be an Exponential(λ) random variable having PDF

$$f_2(y;\, \lambda) = \begin{cases} e^{-y} \text{ for } y > 0 \\ 0 \quad \text{otherwise.} \end{cases}$$

Then for any arbitrary constant A, the MGF of $A - X$ is given by

$$M_{A-X}\,(\omega) = e^{\omega A}\,(1 + \lambda\,\omega)^{-1} \quad \text{for} \quad \omega > -\frac{1}{\lambda}.$$

Proof: It can be proved by simple integration and hence the proof is omitted.

Table 9: Data for illustrative example.

Stress level	Data					
1	10.05	10.59	12.73	12.99	13.71	14.03
	14.34					
2	14.53	14.97	15.37	15.43	15.48	15.60
	15.76	16.18	16.46	16.86	16.90	17.02
	17.36	17.62	18.06	18.31	18.69	18.94
	18.95	22.65	22.89	24.51	25.39	

Table 10: Results of data analysis.

		Type of	μ		θ_1		θ_2		Median	
	r	CI/CRI	LL	UL	LL	UL	LL	UL	LL	UL
95%	30	ACI	–	–	4.46	29.95	2.07	4.93	–	–
		BCI	–	–	7.39	20.70	2.24	4.32	15.47	36.77
		Per. CRI	9.89	9.96	6.54	11.10	4.29	7.27	16.15	36.59
		HPD CRI	9.90	9.96	6.34	10.76	4.15	7.05	14.47	32.68
	20	ACI	–	–	4.46	29.95	1.69	5.70	–	–
		BCI	–	–	7.39	20.70	2.09	5.27	15.47	36.77
		Per. CRI	9.84	9.94	8.15	14.78	5.34	9.69	16.16	37.74
		HPD CRI	9.84	9.93	8.28	14.84	5.43	9.73	15.10	34.21
99%	30	ACI	–	–	0.45	33.96	1.62	5.38	–	–
		BCI	–	–	6.27	23.62	2.07	4.83	14.32	51.85
		Per. CRI	9.88	9.97	5.86	12.10	3.84	7.94	15.12	50.11
		HPD CRI	9.89	9.97	5.70	11.68	3.73	7.66	14.35	43.59
	20	ACI	–	–	0.45	33.96	1.05	6.33	–	–
		BCI	–	–	6.27	23.62	1.90	5.90	14.32	51.85
		Per. CRI	9.83	9.95	7.30	15.52	4.78	10.17	15.14	48.41
		HPD CRI	9.83	9.94	7.59	15.55	4.97	10.19	13.74	43.31

Corollary .1. Let X be a Gamma(α, λ_1) random variable, Y be an Exponential(λ_2) random variable and they are independently distributed. Then for any arbitrary constant A, the MGF of $A + X - Y$ is

$$M_{A+X-Y}(\omega) = e^{\omega A}\,(1-\lambda_1\,\omega)^{-\alpha}\,(1+\lambda_2\,\omega)^{-1} \quad \text{for } -\frac{1}{\lambda_2} < \omega < \frac{1}{\lambda_1}.$$

PROOF: Using Lemmas .3 and .4, it can be proved easily.

Lemma .5. Let X be a Gamma(α, λ_1), Y an Exponential(λ_2) random variable and they are independently distributed. Then the PDF of $X - Y$ is given by

$$f(t;\alpha,\lambda_1,\lambda_2) = \frac{1}{\lambda_1^{\alpha}\,\lambda_2\,\Gamma(\alpha)}\,e^{t/\lambda_2}\int_{\max\{0,t\}}^{\infty} z^{\alpha-1}e^{-(\frac{1}{\lambda_1}+\frac{1}{\lambda_2})z}dz \quad \text{for } t \in \mathbb{R}.$$

Proof: It can be proved using transformation of variable.

Proof of Theorem 3.1:

Using the lemma .1, we get

$$E[e^{\omega\widehat{\theta_1}}|N=1] = \frac{1}{\theta_1\left(1-e^{-\frac{\tau-\mu}{\theta_1}}\right)}\int_{\mu}^{\tau} e^{\omega(t_1-nt_1+(n-1)\tau)-\frac{1}{\theta_1}(t_1-\mu)}dt_1$$

$$= \frac{e^{-\frac{1}{\theta_1}(\tau-\mu)}}{\theta_1\left(1-e^{-\frac{\tau-\mu}{\theta_1}}\right)}\int_{\mu}^{\tau} e^{\left(\omega n-\omega+\frac{1}{\theta_1}\right)(t_1-\tau)}dt_1$$

$$= \frac{e^{-\frac{1}{\theta_1}(\tau-\mu)}}{\theta_1\left(1-e^{-\frac{\tau-\mu}{\theta_1}}\right)} \times \frac{e^{\left(\omega n-\omega+\frac{1}{\theta_1}\right)(\tau-\mu)}-1}{\omega n-\omega+\frac{1}{\theta_1}}. \tag{12}$$

Using the lemma .1, we get for $i = 2(1)\overline{r-1}$

$$E[e^{\omega\widehat{\theta_1}}|N=i] = \frac{i!}{\theta_1^i\left(1-e^{-\frac{\tau-\mu}{\theta_1}}\right)^i}$$
$$\times \int_{\mu}^{\tau}\int_{t_1}^{\tau}\cdots\int_{t_{i-2}}^{\tau}\int_{t_{i-1}}^{\tau} e^{\frac{\omega}{i}(\sum_{j=1}^{i}t_j-nt_1+(n-i)\tau)-\frac{1}{\theta_1}\sum_{j=1}^{i}(t_j-\mu)}dt_i\cdots dt_1$$
$$= \frac{i!e^{-\frac{i}{\theta_1}(\tau-\mu)}}{\theta_1^i\left(1-e^{-\frac{\tau-\mu}{\theta_1}}\right)^i}$$
$$\times \int_{\mu}^{\tau}\int_{t_1}^{\tau}\cdots\int_{t_{i-2}}^{\tau}\int_{t_{i-1}}^{\tau} e^{-\left(\frac{\omega n}{i}-\frac{\omega}{i}+\frac{1}{\theta_1}\right)(t_1-\tau)-\left(\frac{1}{\theta_1}-\frac{\omega}{i}\right)\sum_{j=2}^{i}(t_j-\tau)}dt_i\cdots dt_1$$
$$= \left(\frac{1}{\theta_1}-\frac{\omega}{i}\right)^{-1}\int_{\mu}^{\tau}\cdots\int_{t_{i-2}}^{\tau} e^{-\left(\frac{\omega n}{i}-\frac{\omega}{i}+\frac{1}{\theta_1}\right)(t_1-\mu)-\left(\frac{1}{\theta_1}-\frac{\omega}{i}\right)\sum_{j=2}^{i-1}(t_j-\tau)}$$
$$\times\left\{e^{-\left(\frac{1}{\theta_1}-\frac{\omega}{i}\right)(t_{i-1}-\tau)}-1\right\}dt_{i-1}\cdots dt_1$$
$$\vdots$$
$$= \frac{e^{-\frac{i}{\theta_1}(\tau-\mu)}}{\left(1-e^{-\frac{1}{\theta_1}(\tau-\mu)}\right)^i}\sum_{j=0}^{i-1}(-1)^{i-j-1}\binom{i}{j+1}\frac{e^{\left\{\frac{\omega}{i}(n-j-1)+\frac{1}{\theta_1}(j+1)\right\}(\tau-\mu)}-1}{\left(1-\frac{\omega\theta_1}{i}\right)^{i-1}\left(1+\frac{\omega(n-j-1)\theta_1}{j(i+1)}\right)}. \tag{13}$$

Hence using (7), (12), and (13), we have

$$M_1(\omega|A) = \sum_{i=1}^{r-1}\sum_{j=0}^{i-1} \frac{(-1)^{i-j-1}}{\sum_{k=1}^{n-1} p_k}\binom{n}{i}\binom{i}{j+1} e^{-\frac{n}{\theta_1}(\tau-\mu)} \frac{e^{\left\{\frac{\omega}{i}(n-j-1)+\frac{1}{\theta_1}(j+1)\right\}(\tau-\mu)}-1}{\left(1-\frac{\omega\theta_1}{i}\right)^{i-1}\left(1+\frac{\omega(n-j-1)\theta_1}{i(j+1)}\right)}$$
$$= \sum_{i=1}^{r-1}\sum_{j=0}^{i-1} c_{ij}\frac{e^{\omega\tau_{ij}}}{\left(1-\frac{\theta_1\omega}{i}\right)^{i-1}\left(1+\frac{(n-j-1)\theta_1\omega}{(j+1)i}\right)}$$
$$-\sum_{i=1}^{r-1}\sum_{j=0}^{i-1} d_{ij}\frac{1}{\left(1-\frac{\theta_1\omega}{i}\right)^{i-1}\left(1+\frac{(n-j-1)\theta_1\omega}{(j+1)i}\right)}.$$

where τ_{ij}, c_{ij} and d_{ij} are defined in (9). Now using Lemmas .3, .5 and Corollary .1, we have (8) and this completes the proof of the Theorem 3.1.

Proof of Theorem 3.2

CMGF of $\widehat{\theta}_2$ can be expressed as

$$E[e^{\omega\widehat{\theta}_2}|1 \le N \le r-1] = \sum_{i=1}^{r-1} E[e^{\omega\,\widehat{\theta}_2}|N=i] \times P(N=i|1 \le N \le r-1).$$

Using Lemma .2, for $i = 1, 2, \cdots, r-1$

$$E[e^{\omega\,\widehat{\theta}_2}|N=i] = \frac{(n-i)!}{(n-r)!\,\theta_2^{r-i}}\, e^{-\left(\frac{\omega}{r-i}-\frac{1}{\theta_2}\right)(n-i)\tau}$$
$$\times \int_\tau^\infty\int_{t_{i+1}}^\infty \cdots \int_{t_{n-1}}^\infty e^{-\sum_{j=i+1}^{r-1}\left(\frac{1}{\theta_2}-\frac{\omega}{r-i}\right)t_{j:n}-\left(\frac{1}{\theta_2}-\frac{\omega}{r-i}\right)(n-r+1)t_{r:n}} dt_n \cdots dt_{i+1}$$
$$= \frac{1}{\left(1-\frac{\theta_2\,\omega}{r-i}\right)^{r-i}}.$$

Therefore

$$E[e^{\omega\widehat{\theta}_2}|1 \le N \le r-1] = \sum_{i=1}^{r-1} \frac{1}{\left(1-\frac{\theta_2\,\omega}{r-i}\right)^{r-i}} \times \frac{p_i}{\sum_{k=1}^{r-1} p_k}.$$

Using the Lemma .3, we have the Theorem 3.2.

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