Analysis of load-sharing systems using a model based on piecewise linear functions

Ayon Ganguly

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Department of Mathematics
Indian Institute of Technology Guwahati

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Outline

Introduction

What are Load-Sharing Systems?

Definition
Systems with multiple components sharing operational load

Dynamic Stress
When one component fails, the surviving components must take on the extra load.

Consequence
The redistribution of stress increases the failure rate of the remaining operating components.

Real-World Examples

Engineering
Multi-motor systems, power generators running in parallel, cables with multiple strands.

Software & Computing
Server farms distributing network traffic; if one server crashes, others experience higher traffic loads.

Medical & Biological
Paired organs (e.g., eyes, kidneys) where the failure or impairment of one increases the burden on the other.

Model Formulation

Form of Data (1/2)

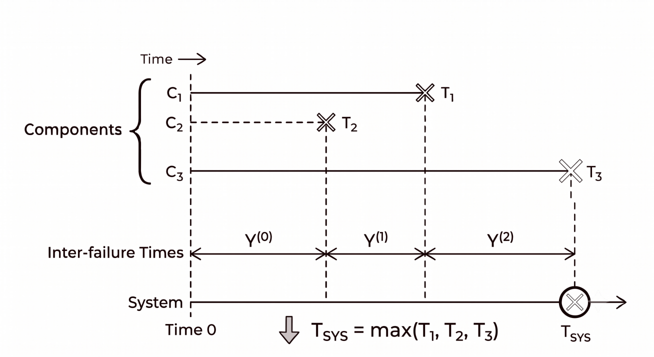


Figure 1: Illustration — A parallel system with 3 components

Form of Data (2/2)

Assumptions:

- J -component parallel systems
- Constant total load on each system
- Failed components of systems are not replaced or repaired.

Form of Data on n Systems

	Inter-failure times			
System 1	$y_1^{(0)}$	$y_1^{(1)}$	$\cdots$	$y_1^{(J-1)}$
System 2	$y_2^{(0)}$	$y_2^{(1)}$	$\cdots$	$y_2^{(J-1)}$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
System n	$y_n^{(0)}$	$y_n^{(1)}$	$\cdots$	$y_n^{(J-1)}$

Model for $Y^{(j)}$'s

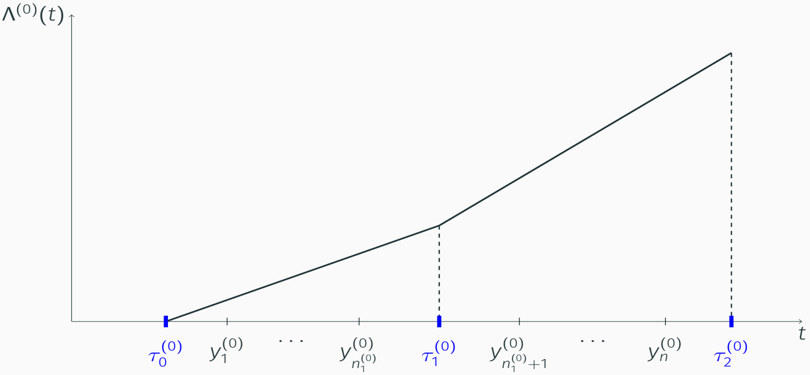
$Y^{(0)}$	$Y^{(1)}$	$Y^{(2)}$		$Y^{(J-1)}$
No. of working components J	No. of working components $J-1$	No. of working components $J-2$		No. of working components 1
$\uparrow$	$\uparrow$	$\uparrow$		$\uparrow$
$U_1^{(0)}$	$U_1^{(1)}$	$U_1^{(2)}$		$U_1^{(0)}$
$U_2^{(0)}$	$U_2^{(1)}$	$U_2^{(2)}$		$\uparrow$
$\vdots$	$\vdots$	$\vdots$		min
$U_{J-2}^{(0)}$	$U_{J-2}^{(1)}$	$U_{J-2}^{(2)}$		
$U_{J-1}^{(0)}$	$U_{J-1}^{(1)}$	$\uparrow$		
$U_J^{(0)}$	$\uparrow$	min		
$\uparrow$	min			
min				

Model for $Y^{(j)}$

- $Y^{(j)} = \min\left\{U_1^{(j)}, U_2^{(j)}, \ldots, U_{J-j}^{(j)}\right\}$
- $U_1^{(1)}, U_2^{(1)}, \ldots, U_{J-1}^{(1)}$: Latent lifetimes of the operating components after the failure of one component
- Latent lifetimes are independent and identically distributed
- $H^{(j)}(\cdot)$: Cumulative hazard function of latent lifetimes
- $P\left(Y^{(j)} > y\right) = \exp\left\{-(J-j)H^{(j)}(y)\right\}, \quad y > 0$

PLA-based Model

- We approximate $H^{(j)}(\cdot)$ using piecewise linear function
- Start with $H^{(0)}(\cdot)$



PLA for $H^{(0)}(\cdot)$

Cut Points

- Cut points: $\xi^{(0)} = \left\{\tau_0^{(0)}, \tau_1^{(0)}, \ldots, \tau_N^{(0)}\right\}$
- $\tau_0^{(0)} < \tau_1^{(0)} < \ldots < \tau_N^{(0)}$
- $\tau_0^{(0)} \leq \min\left\{y_1^{(0)}, \ldots, y_n^{(0)}\right\}$
- $\tau_N^{(0)} \geq \max\left\{y_1^{(0)}, \ldots, y_n^{(0)}\right\}$

PLA of $H^{(0)}(\cdot)$

$$\Lambda^{(0)}(t) = \sum_{k=1}^{N}(a_k + b_k t)\mathbf{1}_{[\tau_{k-1}^{(0)}, \tau_k^{(0)})}(t)$$

where $\mathbf{1}_A(t)$ is the indicator function.

PLA of $H^{(0)}(\cdot)$

Assumptions on $\Lambda^{(0)}(\cdot)$

- $\Lambda^{(0)}(\tau_0^{(0)}) = 0$
- $\Lambda^{(0)}(\cdots)$ is continuous

Reduction in number of parameters

$$a_1 = -b_1\tau_0^{(0)} \quad \text{and} \quad a_k = \sum_{\ell=1}^{k-1} b_\ell(\tau_\ell^{(0)} - \tau_{\ell-1}^{(0)}) - b_k\tau_{k-1}^{(0)}$$

for $k = 1, 2, 3, \ldots, N$.

PLA of $H^{(j)}(\cdot)$

Cut points

- Cut points: $\xi^{(j)} = \left\{\tau_0^{(j)}, \tau_1^{(j)}, \ldots, \tau_N^{(j)}\right\}$
- $\tau_0^{(j)} < \tau_1^{(j)} < \ldots < \tau_N^{(j)}$
- $\tau_0^{(j)} \leq \min\left\{y_1^{(j)}, \ldots, y_n^{(j)}\right\}$ and $\tau_N^{(j)} \geq \max\left\{y_1^{(j)}, \ldots, y_n^{(j)}\right\}$

PLA of the CHF $H^{(j)}$

$$\Lambda^{(j)}(t) = \gamma_j \sum_{k=1}^{N} \left[\sum_{\ell=1}^{k-1} b_\ell(\tau_\ell^{(j)} - \tau_{\ell-1}^{(j)}) + b_k(t - \tau_{k-1}^{(j)})\right] \mathbb{1}_{[\tau_{k-1}^{(j)}, \tau_k^{(j)})}(t)$$

Assumptions

- $1 < \gamma_1 < \gamma_2 < \cdots < \gamma_{J-1} \quad \text{and} \quad 0 < b_1 < b_2 < \cdots < b_N$
- Ensures increasing HR in successive failure stages

Inference

Log-likelihood Function

The log-likelihood function becomes

$$l(\boldsymbol{\theta}) = \sum_{k=1}^{N} \left[\left(\sum_{j=0}^{J-1} n_k^{(j)} \right) \ln b_k - \left(\sum_{j=0}^{J-1} (J-j)\gamma_j T_k^{(j)} \right) b_k \right] + n \sum_{j=0}^{J-1} \ln \gamma_j$$

where

$$I_k^{(j)} = \{ i : y_i^{(j)} \in [\tau_{k-1}^{(j)}, \tau_k^{(j)}) \}$$

$$n_k^{(j)} = |I_k^{(j)}|$$

$$T_k^{(j)} = \sum_{i \in I_k^{(j)}} (y_i^{(j)} - \tau_{k-1}^{(j)}) + \left(n - \sum_{\ell=1}^{k} n_\ell^{(j)} \right) (\tau_k^{(j)} - \tau_{k-1}^{(j)})$$

Optimization of the Log-Likelihood

Expressing b_k in terms of γ_j's

$$\frac{\partial l(\boldsymbol{\theta})}{\partial b_k} = 0 \implies b_k = b_k(\boldsymbol{\gamma}) = \frac{\sum_{j=0}^{J-1} n_k^{(j)}}{\sum_{j=0}^{J-1}(J-j)\gamma_j T_k^{(j)}}$$

Profile Log-Likelihood

$$\tilde{l}(\boldsymbol{\gamma}) = \sum_{k=1}^{N}\left[\left(\sum_{j=0}^{J-1} n_k^{(j)}\right)\left\{\ln\left(\sum_{j=0}^{J-1} n_k^{(j)}\right) - \ln\left(\sum_{j=0}^{J-1}(J-j)\gamma_j T_k^{(j)}\right)\right\}\right] + n\sum_{j=0}^{J-1}\ln\gamma_j$$

Maximum Likelihood Estimates

- Optimize $\tilde{l}(\boldsymbol{\gamma})$ numerically to obtain MLEs $\hat{\gamma}_1, \ldots, \hat{\gamma}_{J-1}$.
- MLE of b_k: $\hat{b}_k = b_k(\hat{\gamma}_1, \ldots, \hat{\gamma}_{J-1})$

Confidence Interval

One can construct Bootstrap confidence interval

Choosing Cut-Points for PLA-Based Model

The problem

- Need to choose number/position of cut-points
- Close approximation vs. overfitting.
- Too many cut-points increase computational expense

Approach

- Fix number of cut-points
- Take different choices of cut-points (position)
- Calculate maximum log-likelihood function
- Choose the cut-points that maximizes maximum log-likelihood

Implementation Notes

- Avoid making any interval very small

Robustness Study

Objective

To assess the robustness of the PLA-based model in fitting data obtained from other parent models

Approach

- Generate data from other reasonable model
- PLA-based model is fitted to the generated data
- Model fitting quality is assessed using AIE

Absolute Integrated Error

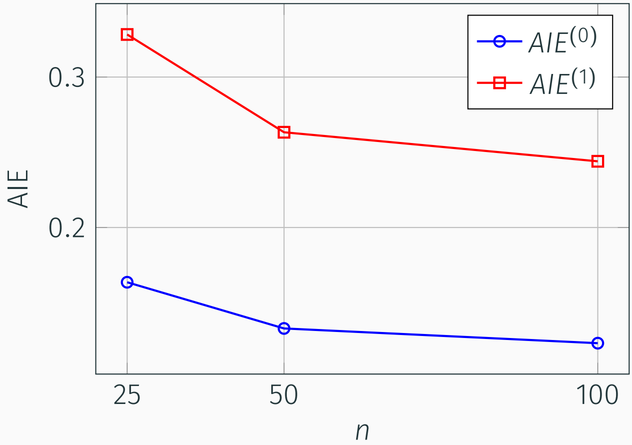
$$AIE^{(j)} = \frac{1}{R}\sum_{k=1}^{R} \frac{1}{y_M^{(j)} - y_m^{(j)}} \int_{y_m^{(j)}}^{y_M^{(j)}} \left| H_{TGP}^{(j)}(t) - \widehat{H}_{PLA}^{(j)}(t) \right| dt$$

- $H_{TGP}^{(j)}(\cdot)$: True CHF of lifetimes between failures
- $\widehat{H}_{PLA}^{(j)}(\cdot)$: PLA-based estimates

Simulation Scenario and AIE Trend

Two-Component Load-Sharing System

- **Initial Phase:** Components are i.i.d. Wei(1.5, 1)
- **After First Failure:** Surviving component has a Wei(1.5, 3)



Real Data Application (1/3)

Dataset

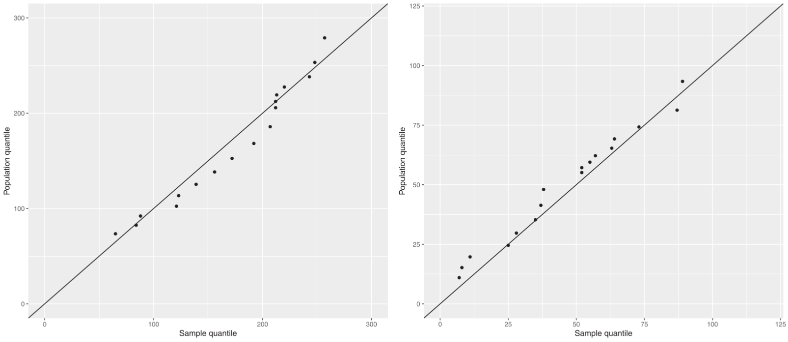
- Failure times of a two-motor load-sharing system
- Motor A and B — both are identical
- Sample size = 18
- When both motors A and B are in working condition, the total load on the system is equally shared between them.
- When one of the motors fails, the total load shifts to the operational motor.

Implementation

- Three cut-points for the PLA-based model
- Position of the cut-points are decided based on the dataset

Parameters Estimates

Parameter	MLE	Std. error	P. bootstrap
γ_1	4.271	1.1901	(3.075, 8.028)
b_1	0.003	0.0008	(0.002, 0.006)
b_2	0.013	0.0039	(0.006, 0.021)



(a) Q-Q plot for $Y^{(0)}$ (b) Q-Q plot for $Y^{(1)}$

Figure 2: Q-Q plot

Conclusion

Summary and Future Work

Summary

- Developed a semi-parametric, data-driven approach using piecewise linear cumulative hazards.
- Derived exact reliability functions and complete inferential procedures (MLE, asymptotic normality).
- Showed robust performance under unknown hazard geometries.

Future Directions

- Extending the framework to general k-out-of-n systems.

Thank You

aganguly@iitg.ac.in
https://ayonganguly.github.io/