Indian Institute of Technology Guwahati
Statistical Inference (MA682)
Problem Set 03

1. Let $X_1,\, X_2,\, \ldots,\, X_n$ be a random sample form $\mathrm{U}(\theta_1,\, \theta_2)$, $-\infty < \theta_1 < \theta_2 < \infty$. Find moment estimator of $(\theta_1,\, \theta_2)$.

2. A sample $(X_1, \ldots,\, X_{10})$ is drawn from a distribution with a probability density function
$$\frac{1}{2}\left(\frac{1}{\theta}e^{-x/\theta} + \frac{1}{10}e^{-x/10}\right),\ \ 0 < x < \infty.$$
The sum of all 10 observations equals 150. Estimate θ by the method of moments.

3. Let $X_1,\, \ldots,\, X_n$ be a random sample form the probability density function
$$f(x;\, \theta) = \frac{1}{2}e^{-|x-\theta|} \quad \text{for } x \in \mathbb{R},$$
where $\theta \in \mathbb{R}$. Find the maximum likelihood estimator of θ.

4. Estimate the unknown parameter θ from a sample $\{3, 3, 3, 3, 3, 7, 7, 7\}$ drawn from a discrete distribution with the probability mass function
$$P(3) = \theta,\ P(7) = 1 - \theta.$$
Obtain MME and MLE of θ.

5. Let $X_1,\, \ldots,\, X_n$ be a random sample form the probability mass function
$$f(x;,\, \theta) = \begin{cases} \frac{1-\theta}{2} & \text{if } x = 1 \\ \frac{1}{2} & \text{if } x = 2 \\ \frac{\theta}{2} & \text{if } x = 3 \\ 0 & \text{otherwise.} \end{cases}$$
Find the maximum likelihood estimator of $\theta \in (0,\, 1)$.

6. Let $X_1,\, X_2,\, \ldots,\, X_n$ be a sample from probability mass function
$$P(X = k) = \begin{cases} \dfrac{1}{N} & \text{if } k = 1,\, 2,\, \ldots,\, N \\ 0 & \text{otherwise,} \end{cases}$$
where N is a positive integer. Find the maximum likelihood estimator of N.

7. Suppose we want to estimate the number of fishes in a pond. The following procedure is followed to perform the estimation. First M fishes are caught from the pond, tagged and returned to the

pond. Next n fishes are caught at random out of which x fishes are tagged. Based on this data, find the maximum likelihood estimator of the total number of fishes present in the pond. Hint: You should not differentiate the likelihood function with respect to N, as N is integer valued and $N \geq \max\{M, n\}$.

8. Let $X_1, X_2, \ldots, X_n$ be a random sample on the lifetime of an integrated circuit. Let the lifetime of the integrated circuit has the probability density function

$$f(x, \theta) = \begin{cases} 2\lambda x e^{-\lambda x^2} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\tau > 0$ be a known time, and X be the number of integrated circuits that fail before τ. Find the maximum likelihood estimator of the variance of X.

9. Suppose that $X_1, X_2, X_3, X_4 \overset{i.i.d.}{\sim} N(0, \sigma^2)$, where $\sigma > 0$ is unknown parameter. Consider the following estimators:

$$T_1 = X_1^2 - X_2 + X_4, \quad T_2 = \frac{1}{3}\left(X_1^2 + X_2^2 + X_4^2\right), \quad T_3 = \frac{1}{4}\sum_{i=1}^{4} X_i^2, \quad \text{and} \quad T_4 = \frac{1}{3}\sum_{i=1}^{4}\left(X_i - \overline{X}\right)^2.$$

(a) Is T_i UE for σ^2 for $i = 1, 2, 3, 4$?

(b) Among estimators T_1, T_2, T_3 and T_4 for σ^2, which one has the smallest MSE?

10. Suppose that $X_1, X_2 \overset{i.i.d.}{\sim} N(0, \sigma^2)$, where $\sigma > 0$ is unknown parameter. Consider the following estimators:

$$T_5 = \frac{1}{2}\left|X_1 - X_2\right|.$$

Is T_5 UE for σ? If not, propose an UE for σ based on T_5. Calculate the MSE of T_5 as an estimator of σ.

11. Let $X_1, X_2, \ldots, X_n$ be a RS with a common PDF

$$f(x) = \frac{1}{\sigma}\exp\left[-\frac{x-\mu}{\sigma}\right] I_{(\mu, \infty)}(x).$$

Denote $U = \sum_{i=1}^{n}\left(X_i - X_{(1)}\right)$ and let $V = cU$ be an estimator of σ, where $c > 0$ is a constant.

(a) Find the MSE of V. Then, minimize the MSE with respect to c. Call this latter estimator W, which has smallest MSE.

(b) How do estimators W and $\frac{U}{n-1}$ compare relative to their respective bias and the MSE?

12. Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} Bernoulli(p)$, where $p \in (0, 1)$ is unknown parameter. Show that there is no UE for the parametric functions

(a) $\tau(p) = \dfrac{1}{p(1-p)}$,

(b) $\tau(p) = \dfrac{1}{p(1-p)^2}$.

13. Suppose $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$, where $\mu \in \mathbb{R}$ and $\sigma > 0$ are unknown parameters. Denote, for $n \geq 1$,

$$T_n = \frac{2X_1 + 4X_2 + 6X_3 + \ldots + 2nX_n}{n(n+1)}.$$

(a) Evaluate $E(T_n)$ and $Var(T_n)$ for all $n \geq 1$.

(b) Show that $\{T_n : n \geq 1\}$ is consistent for μ.

(c) Is $\max\{0, T_n\}$ consistent for μ?